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**Coping with the Planning of Rewards Supply in Loyalty Reward Programs– A  
Mathematical Formulation Model**

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**Abstract**

This paper addresses the problem of planning the supply of rewards, a key operational management issue in loyalty reward programs (LRPs) operations. We discuss the problem in the context of supply chain management and propose a mathematical model that seeks to maximize the LRP firm value creation, subject to satisfying budget and capacity limitations as well as taking into account demand uncertainties and various liability control strategies. A solution procedure based on stochastic programming is developed and its implementation is discussed.

**1. Introduction**

Loyalty Reward programs (LRPs) are marketing programs that offer consumers incentives or rewards based on their repeated purchase behavior with the expectation that these incentives or rewards will serve as a motivation to them to continue purchasing a product or a service. These programs are considered as an important component of customer relationship management strategy to target long-term customer profitability (Liu, 2007; Meyer-Waarden, 2008; Jain and Singh, 2002). LRPs exist today across a spectrum of industries such as travel, hotel, retail,

telecommunication, banking, gasoline, gaming, and entertainment. In recent years, there has been a general recognition in the industry of a need for more sophisticated loyalty-based systems capable of responding to long-term competitive threats such as retail overcapacity, spending on mass advertising, consumer attrition issues, etc. Two new trends of LRPs development have been noticed. In one hand, some of the LRP service providers have replaced the traditional LRP host firms (e.g., airline companies or retail firms) to become LRP hosts themselves and treat LRP activities as their core business. For these loyalty-based service companies, the primary source of their revenue comes from selling miles/points to their accumulation partners. In the other hand, in order to compete effectively and continue to contribute to value growth, many existing LRPs have been restructured or expanded in scope to partner with other firms to offer new products and/or services. For example, Aeroplan®, Canada's premier loyalty program, was founded in 1984 by an airline company (Air Canada) as an internal marketing program. Since then, Aeroplan has experienced organizational restructure and expansion several times. Now the program is owned and operated by Group Aeroplan Inc., a loyalty-service-oriented company, and it is a joint venture with more than 60 partners representing more than 100 brands. Air Miles®, a primary competitor of Aeroplan® founded in 1992, is owned and operated by another loyalty-service-oriented company named Alliance Data Loyalty and Marketing Services. Air Miles offers its 9.5 million Canadian members more than 800 different leisure, entertainment, merchandise, travel and other lifestyle rewards when they shop at one of more than 120 brand-name sponsors of the program<sup>1</sup>.

Unlike other short-term marketing programs, developing or joining in a LRP requires a firm to make a long-time commitment and costly investments. Moreover, with the increasing

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<sup>1</sup> <https://www.airmiles.ca/arrow/SponsorDirectory> (accessed on Aug. 16, 2009)

economic impacts of these programs, the management complexity faced by LRP enterprises has also increased. One of the critical operational issue faced by LRPs is that of efficiently and effectively planning the supply of rewards (and points) in order to achieve management goals such as meeting customer demand, improving customer satisfaction, lowering operational costs or generating higher profits, while taking into account both internal dynamics and external uncertainties (e.g., changes in the settings of system parameters, demand uncertainties and competition threats). As far as we know, this issue has not yet to be addressed in the academic literature. The majority of the existing papers are limited in their coverage to marketing-oriented LRP management issues such as design and implementation of LRPs (e.g. Kada and Kotanko, 2001; Shugan, 2005; Berman, 2006), short- or long-term impact of LRPs on consumer purchase behaviour, attitude, and decision (e.g., Sharp and Sharp 1997; Zhang et al., 2000; Lewis, 2005; Liu, 2007; Meyer-Waarden, 2008), and influence of LRPs on a firm's market competition (e.g. Caminal and Matutes, 1990; Kim et al., 2001). In recent years, a few studies have started to investigate specific operational issues in LRP management. In these studies analytical models have been developed to support planning and management decision making. For example, Kim *et al.* (2004) focused on an LRP's promotion function to examine the adoption and design of an LRP in the context of capacity management. Diaby and Nsakanda (2008) considered the LRP as a dynamic system where LRP members are grouped into membership tiers and migrate from one tier to another according to their accumulation and redemption activities. The authors developed a quantitative model in which the accumulation and redemption of points follow a stochastic process to determine the "breakage rate"<sup>2</sup>. Nsakanda *et al.* (2009) proposed a predictive model of redemption and liability in LRPs. They proposed an 8-step procedure to predicting the state

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<sup>2</sup> "Breakage rate" refers to accumulated points that end up not being redeemed by LRP members. In the LRP industry, breakage rate is claimed as part of the host's revenue in the financial statements (e.g., Aeroplan annual report 2007, page.24. <http://www.groupeaeroplan.com/pages/invReports.php>)

variables of accumulation, redemption, and liability, respectively. Labbi and Berrospi (2007) explored the optimization of marketing planning and budgeting in LRPs using Markov decision processes. Enabled by customer behavioral, financial, and demographic information stored in LRPs, a three-step methodology is proposed to assist in assessing long-term customer value and identifying which marketing actions are the most effective in improving customer loyalty and hence increasing revenue. Although relevant to the management of LRPs, these studies do not directly address the issue of efficiently and effectively planning the supply of rewards (and points).

We address in this paper the problem of aggregate planning of the supply of rewards under several management concerns such as LRP value creation, demand uncertainties, budget and capacity limitations and points-liability control. Points-liability<sup>3</sup> is widely recognized in the LRP industry as a risk indicator for a firm's future LRP operations. It represents the value of future redemption obligation of points earned by LRP members. Each time when an LRP member earns points, these points will be added into the LRP firm's liability account and remain in this account until the member redeems the points for a reward. A higher or lower liability has negative impacts on LRP operations, such as higher redemption uncertainties, hyperinflation and devaluation of points, or higher redemption costs. Therefore, maintaining points-liability within a range that will most benefit LRP firms is crucial in LRP management. Our focus is on the most popular types of LRPs in practice. Generally speaking, in this type of LRPs, there are three key players: LRP members, LRP host and LRP partner(s). LRP members refer to end consumers who own a member account and/or a membership card. LRP host refers to the firm that owns or manages the program as a profit center. LRP partners refer to business entities other than the host firm, who join the program to offer accumulation and/or redemption options to LRP members.

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<sup>3</sup> D. Hofer, 2008. Reinforcing the value of frequent flyer miles. Loylogic Inc. White paper. [www.loylogic.com](http://www.loylogic.com)

Each time LRP members purchase products or services from an LRP host or LRP partners, the members' purchase information will be recorded in a database in a form (e.g. amount of points or miles) that can be used later as the basis to calculate the amount and type of rewards that the members can obtain from the LRP host or partners. The LRP host focal business is to provide a brand-name LRP service to LRP members through partnership in both redemption and accumulation operations. There are commonly more than one partner involved in either redemption or accumulation operations to provide goods and/or service in different categories to LRP members. Points are used as the single universal reward media to record customer purchase activities and relate them to different reward categories. We discuss the problem in the context of supply chain management and propose a mathematical model that seeks to maximize the LRP host value creation, subject to satisfying budget and capacity limitations as well as taking into account demand uncertainties and various liability control strategies. We develop a solution procedure based on stochastic programming and discuss its implementation.

The remainder of the paper is organized as follows. First, we present a conceptual model in the following section which describes LRP systems as supply chains. We then describe the rewards-supply planning problem and propose a mathematical model in section 3. Next, we discuss the solution methodology and its implementation in section 4. Finally, we conclude with a discussion of key characteristics and managerial applicability of our model in section 5.

## **2. A Conceptual Model of LRP System –“Rewards-Points” Supply Chain**

LRP systems can be viewed as “rewards-points” supply chains (RSCs) where a host firm runs the LRP as a focal business that seeks to maximize the value created through “rewards & points” business. LRP partners participate in the system by providing accumulation and/or

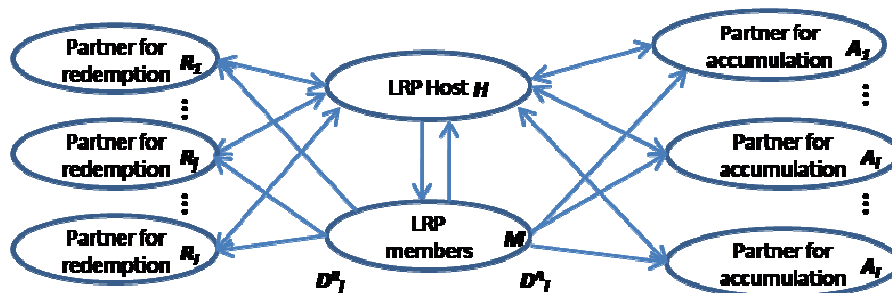
redemption services to fulfill consumer requirements for points and rewards. Furthermore, through this “rewards-points” supply chain, LRP partners also compete with other non-LRP firms in the market to gain additional revenue. In comparison to traditional supply chains (TSCs), there are similarities and differences.

Generally speaking, a TSC consists of multiple independent business entities such as vendors, manufacturers, distributors, retailers, etc. Similarly, in a RSC, LRP host firm, LRP redemption partners, LRP accumulation partners are the multiple independent business entities involved. The revenue flow is the same in both TSCs and RSCs in the sense that the revenue is created by end consumers and shared among all the business entities in the system. In the TSC, the products provided by the business entities are the same, either branded or in the same good/service categories; while in the RSC, rewards in different good/service categories are provided by redemption partners and points are a special type of product provided by LRP host and delivered by accumulation partners. End consumers’ demand relate to products only in the TSC, while in the RSC, end consumers’ demand relate to both points and rewards. With regard to system structure, the TSC is mainly sequential-based. Following the production flow, the business entities involved are operated sequentially. The downstream entities play a key role in the interaction with end consumers. In contrast, the structure of RSC is parallel-based. The LRP host firm is at the center of the system. The redemption and accumulation partners are the LRP host’s multiple channel partners and operate independently. All these entities in the RSC interact with end consumers (i.e. LRP members) directly. Taken in this sense, the structure of RSCs is very similar to decentralized multi-channel supply chain. Another key difference between the TSC and the RSC is that main costs and revenues are generated in a different time order. In the TSC, costs associated with production occur first, and then revenues are generated through

selling products. In the RSC, revenues are generated through selling points first, and then costs associated with rewards occur later when LRP members redeem their points for rewards.

The common characteristics between TSCs and RSCs motivate us to apply TSC models to study an LRP system. Meanwhile, the unique features of RSCs allow us to explore the special problems associated with LRP operations. Based on this point of view, a conceptual model of LRP systems can be described as shown in Figure 1.

**Figure 1: A Conceptual Model of LRP Systems**



In this rewards-points supply chain, LRP host  $H$  has multiple commercial partners at both redemption (i.e.,  $R_j, j = 1, \dots, J$ ) and accumulation (i.e.,  $A_i, i = 1, \dots, I$ ) sides. A LRP member who collects and redeems points in this system has two types of demand: point-accumulation demand and point-redemption demand. LRP members' demand on accumulation,  $D_i^A, (i = 1, \dots, I)$ , drives the points business between LRP host  $H$  and LRP partner  $A_i$  (i.e.,  $H$  sells points to  $A_i$ ). Meanwhile, members' demand on redemption,  $D_j^R, (j = 1, \dots, J)$ , drives the rewards business between the LRP host  $H$  and the LRP partner  $R_j$  (i.e.,  $H$  purchases rewards from  $R_j$ ). How does this rewards-points supply chain create values for all the entities involved in the system? At the accumulation side, when a LRP member, say  $M$ , purchases products from a LRP accumulation

partner  $A_i$ , the value from points business is created. For the LRP member,  $M$ , the value obtained is through points that are collected. For the LRP partner,  $A_i$ , the value refers to the revenue generated by providing accumulation services (i.e., give out different amount of points based on the purchases of the LRP member). Meanwhile, as the LRP partner  $A_i$  does not own points,  $A_i$  has to purchase them from LRP host  $H$  - the issuer of points. By selling points to LRP partner  $A_i$ , LRP host  $H$  shares the revenue that  $A_i$  gains. The shares of revenues from all accumulation partners are the value that LRP host  $H$  obtains through the points business. At the redemption side, when a LRP member, say  $M$ , requests a reward, the LRP host  $H$  will purchase it from LRP partner  $R_j$  for the member. Meanwhile, the LRP host  $H$  charges the LRP member  $M$  a certain amount of points for the requested reward. For the LRP host  $H$ , the value obtained is the difference between the market value<sup>4</sup> of the reward and the actual cost that the LRP host  $H$  spends to purchase the reward. In the same process, the value obtained by the LRP member  $M$  is the reward and the value that the LRP partner  $R_j$  obtains is the revenue generated by selling the reward to the LRP host  $H$ .

In the remaining of this paper, the overall value that LRP host firm obtain through this rewards-points supply chain is named as the LRP host profitability.

### 3. Problem description and model formulation

Let  $A_i, i=1,2,\dots,I$ , be the LRP accumulation partners in the system,  $q_i^A$  be the LRP accumulation partner  $A_i$  ordering quantity of points,  $D_i^A$  be the LRP members accumulation demand towards the LRP accumulation partner  $A_i$ . Let  $R_j, j=1,2,\dots,J$ , be the LRP redemption partners in system,  $q_j^R$  be the LRP host  $H$ 's ordering quantity of rewards from LRP redemption

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<sup>4</sup> The value of rewards quite often depends on the benchmarks, for example, the retail price of the reward in the market.

partner  $R_j$ ,  $D_j^R$  be the LRP members' redemption demand towards the LRP redemption partner  $R_j$ 's rewards. The problem we are concerned is that of determining, given the ordering quantities of points from LRP accumulation partners,  $A_i$ 's, (i.e.,  $q_i^A$ 's), the LRP host  $H$ 's optimal ordering quantity decisions of rewards from LRP partners  $R_j$ 's (i.e.,  $q_j^R$ 's), in order to maximize the LRP host profitability (as measured by its value creation), subject to the LRP partners  $R_j$ 's capacities on offering rewards, the LRP host  $H$ 's overall budget for purchasing rewards, and LRP host  $H$ 's control on points-liability. Our model is developed based on single-period constrained newsvendor model.

Similarly to the traditional supply chain operations, we assume that the relationships between LRP partners (both for redemption and/or accumulation) and LRP host are primarily governed by contracts. Although many different types of contracts have been discussed in the literature (e.g., Tsay, 1999; Cachon, 2003), we limit this study to the wholesale-price contracts given its fit for LRPs. Under this contract setting, the LRP host  $H$  guarantees each LRP accumulation partner  $A_i$  a wholesale unit price of points,  $w_i^A$ . The LRP accumulation partner  $A_i$  decides on the quantity of points to order ( $q_i^A$ ) during the planning horizon at the given wholesale unit price. The LRP members accumulation demand (in points) towards the accumulation partner  $A_i$  (i.e.,  $D_i^A$ ) is not known with certainty, but follows a known probability distribution. At the end of the planning horizon, if the LRP members accumulation demand towards partner  $A_i$  is higher than the LRP accumulation partner ordering quantity ( $q_i^A$ ), the per unit excess demand is purchased at  $w_i^A$ , the back-order unit price of points. A reasonable

assumption here is that  $w_i^A \geq w_i^A$ . Hence, the LRP Host  $H$ 's profitability (i.e. value creation) function at the accumulation side can be defined as follows (BP-A):

$$\pi_{H(A)}(; D_i^A) = \sum_{i=1}^I \left( w_i^A \times q_i^A + w_i^A \times [D_i^A - q_i^A]_+ \right)$$

The first term in (BP-A),  $(w_i^A \times q_i^A)$  is the revenue obtained through selling points to the accumulation partner  $A_i$ . The second term,  $w_i^A \times [D_i^A - q_i^A]_+$ , denotes the extra revenue that the LRP host  $H$  gains through  $A_i$ 's back-order of points when  $D_i^A$  is higher than  $A_i$ 's ordering quantity  $q_i^A$ .

At the redemption side, each LRP redemption partner ( $R_j$ ) guarantees the LRP host  $H$  a wholesale unit cost of points redeemed,  $w_j^R$ . The LRP host  $H$  decides on the quantity of rewards (in points) to order during the planning horizon ( $q_j^R$ ) from a LRP redemption partner ( $R_j$ ) at the given wholesale discounted unit cost. The LRP members redemption demand (in points) towards the LRP redemption partner  $R_j$  (i.e.,  $D_j^R$ ) is not known with certainty, but follows a known probability distribution. At the end of the planning horizon, if the LRP members redemption demand towards partner  $R_j$ , is higher than the LRP host ordering quantity ( $q_j^R$ ), the excess demand is assumed to be lost and the under-stocking cost is  $v_j^R$  per unit of points. In the contrary, the excess ordering quantity is sold at  $s_j^R$ , the over-stocking unit sale price. Let  $p_j^R$  defines the per point unit value of rewards offered by partner  $R_j$ . Hence, the LRP Host  $H$ 's profitability (i.e., value creation) function at the redemption side can be defined as follows (BP-R):

$$\pi_{H(R)}(q_j^R; D_j^R) = \sum_{j=1}^J (p_j^R \times \min\{q_j^R, D_j^R\} - w_j^R \times q_j^R - v_j^R \times [D_j^R - q_j^R]_+ + s_j^R \times [q_j^R - D_j^R]_+)$$

(BP-R) is a linear function that consists of four components. The first part,  $p_j^R \times \min\{q_j^R, D_j^R\}$  indicates the value of rewards offered by each partner  $R_j$ . The second part,  $w_j^R \times q_j^R$ , indicates the LRP host  $H$ 's purchasing cost of rewards. The third part,  $v_j^R \times [D_j^R - q_j^R]_+$  is the under-stocking cost of rewards. The fourth part,  $s_j^R \times [q_j^R - D_j^R]_+$  is the salvage value of over-stocking rewards.

In our modeling problem, we also made the following additional assumptions: (a) LRP redemption partners have capacity limitations on offering rewards; (b) LRP host has no capacity limitation on issuing points; (c) LRP members accumulation and redemption demands are not known with certainty, but have known probability distributions and both demands are price-independent; (d) one universal static redemption scheme is adopted by all LRP redemption partners; (e) one universal static accumulation scheme is adopted by all LRP accumulation partners; and (f) LRP members accumulation demand will always be met. Assumptions (b) and (f) relate to the unique features of points. As points are a kind of information symbol for recording and counting LRP members' purchase effort in the LRP system, the LRP host  $H$  does not have any "production" related costs and "resource" related capacity limitations on offering points. As such, to a host, there is no capacity limitation on issuing points. Furthermore, unlike tangible products, production and movement of points are not limited by time and physical space. Points are never "stock-out" in the sense that there is no time lag between the production of points and meeting customers' accumulation demand on points. Therefore, members' accumulation demands will always be met. In other words, the LRP host  $H$  allows the LRP

partner  $A_i$  to “back-order” points that are over its initial ordering quantity. Now let’s consider the additional notation shown in Figure 2.

**Figure 2: List of the Overall Notation**

<b>Indices:</b>	
$R_j$	Redemption partners in the LRP system, $j=1,2,\dots,J$
$A_i$	Accumulation partners in the LRP system, $i=1,2,\dots,I$
$H$	Host firm in the LRP system
<b>Decision variables:</b>	
$q_j^R$	Host $H$ 's ordering quantity of rewards from partner $R_j$ .
<b>Parameters:</b>	
$D_i^A$	Members' accumulation demand towards partner $A_i$ .
$q_i^A$	Partner $A_i$ 's ordering quantity of points.
$w_i^A$	Wholesale price per unit of points host $H$ charges partner $A_i$ .
$w_i^A$	Back-order price per unit of points that host $H$ charges partner $A_i$ when accumulation demand is over partner $A_i$ 's ordering quantity.
$p_j^R$	Per point unit value of rewards offered by partner $R_j$ .
$D_j^R$	Members' redemption demand towards partner $R_j$ 's rewards.
$w_j^R$	Wholesale unit price of rewards that LRP partner $R_j$ charges to LRP host $H$
$v_j^R$	Per unit shortage penalty cost of partner $R_j$ 's rewards.
$s_j^R$	Per unit salvage value of partner $R_j$ 's rewards.
$l_0$	Points-liability at the beginning of the targeted time period.
$l$	Points-liability at the end of the targeted time period.
$L_{UB}$	Upper bound of liability control limits for the targeted time period.
$L_{LB}$	Lower bound of liability control limits for the targeted time period.
$Q_j^R$	Partner $R_j$ 's capacity limitation on offering rewards offered by redemption.

$W^R$	Host $H$ 's budget limitation on purchasing rewards.
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Combining (BP-A) and (BP-R), the problem of planning the supply of rewards can be formulated as follows (BP):

$$\begin{aligned} \max \Pi_H(q_j^R; D_j^R, D_i^A) &= \max E[\pi_{H(R)}(q_j^R; D_j^R) + \pi_{H(A)}(; D_i^A)] \\ &= \max \left( \begin{aligned} &\sum_{i=1}^I (w_i^A \times q_i^A) - \sum_{j=1}^J w_j^R \times q_j^R + E \left[ \sum_{i=1}^I w_i^A \times [D_i^A - q_i^A]_+ \right] + \\ &E \left[ \sum_{j=1}^J (p_j^R \times \min\{q_j^R, D_j^R\} - v_j^R \times [D_j^R - q_j^R]_+ + s_j^R \times [q_j^R - D_j^R]_+) \right] \end{aligned} \right) \end{aligned} \quad (1)$$

subject to:

Liability control constraints:

$$L_{LB} \leq \frac{l}{l_0} \leq L_{UB}, \quad (2)$$

$$\text{where } l = l_0 + \sum_{i=1}^I (q_i^A + [D_i^A - q_i^A]_+) - \sum_{j=1}^J (\min\{q_j^R, D_j^R\})$$

Redemption partners' capacity limitations on offering rewards:

$$q_j^R \leq Q_j^R, \text{ for } j = 1, \dots, J \quad (3)$$

LRP host's budget constraint on purchasing rewards:

$$\sum_{j=1}^J (w_j^R \times q_j^R) \leq W^R, \quad (4)$$

Non-negative constraints:

$$q_j^R \geq 0, \text{ for } j = 1, \dots, J \quad (5)$$

In (2),  $\sum_{i=1}^I (q_i^A + [D_i^A - q_i^A]_+)$  is the overall amount of points accumulated during the planning horizon.  $\sum_{j=1}^J (\min\{q_j^R, D_j^R\})$  is the overall amount of points redeemed during the

planning horizon. As discussed in the introduction section, points collected by LRP members are

stored in members' account. To the LRP host  $H$ , these points are liability until they are redeemed by LRP members for rewards. Therefore, the overall liability at the end of the planning horizon ( $l$ ) is equal to the initial liability ( $l_0$ ) at the beginning of the planning horizon plus the overall amount of points collected by members during the planning horizon, and then minus the overall amount of points redeemed by members for rewards during the same planning horizon.  $L_{LB}$  and  $L_{UB}$  are introduced as the liability control parameters. If the LRP host  $H$  plans to reduce the liability, then  $L_{LB}$  and  $L_{UB}$  are set as:  $0 < L_{LB} < L_{UB} < 1$ . If the LRP host  $H$  plans to keep the liability at the same level as before, then  $L_{LB}$  and  $L_{UB}$  are set to:  $L_{LB} = L_{UB} = 1$ . When  $L_{LB}$  and  $L_{UB}$  are set as  $L_{UB} > L_{LB} > 1$ , the LRP host allows the liability to be at a higher level than that in the previous planning horizon, but within a certain range. Constraint (3) indicates that each redemption side partner has a capacity limitation on quantity of rewards offered to LRP host  $H$ . Constraint (4) indicates that LRP host  $H$  has an overall budget limit for purchasing rewards.

#### **4. Problem solving methodology**

Given the demand uncertainties involved in our model (BP), stochastic programming (SP) approaches provide a promising avenue to solve it. The special structure of (BP) lends itself to a specific type of SP models known as a two-stage stochastic linear programming with recourse (2SLPR). The key characteristic of 2SLPR (Birge and Louveaux, 1997; Ruszczyński and Shapiro, 2003) is that decision variables are classified into two stages according to whether they are implemented before or after an outcome of a (vector valued) random variable is observed. In other words, a set of decisions ( $x$ ) are taken in the first stage without full information on the random vector. Later, full information is received on the possible realization of the random vector; then a second stage action (also called recourse action) is taken. The

second stage decisions (y) allow us to model a response to each of the observed outcomes of the random vector, which constitute our recourse. We discuss in this section first the reformulation of the problem as a two-stage stochastic linear programming with recourse and then we describe our solution methodology.

#### 4.1 Problem reformulation as a two-stage stochastic linear programming with recourse

2SLPR is suitable for decision models with single-period randomness and reaction (e.g., newsvendor-based models). (BP) satisfies this characteristic and can be reformulated into 2SLPR format as follows:

(BP-2SLPR)

$$\max \pi_H = \sum_{i=1}^I (w_i^A \times q_i^A) + \sum_{j=1}^J (p_j^R \times q_j^R - w_j^R \times q_j^R) + E_{\xi} [g(x, \omega)] \quad (6)$$

subject to:

$$q_j^R \leq Q_j^R, \text{ for } j=1, \dots, J \quad (3)$$

$$\sum_{j=1}^J (w_j^R \times q_j^R) \leq W^R \quad (4)$$

$$q_j^R \geq 0, \text{ for } j=1, \dots, J \quad (5)$$

where

$$g(x, \omega) = g(q_j^R, D_i^A, D_j^R) = \max \sum_{i=1}^I (w_i^A I_i^{A-}) + \sum_{j=1}^J (-v_j^R I_j^{R-} + s_j^R I_j^{R+} - p_j^R I_j^{R+}) \quad (7)$$

subject to:

$$L_{LB} \leq \frac{l}{l_0} \leq L_{UB} \quad (2)$$

$$\text{where } l = l_0 + \sum_{i=1}^I (q_i^A + I_i^{A-}) - \sum_{j=1}^J (q_j^R - I_j^{R+})$$

$$I_i^{A+} - I_i^{A-} = q_i^A - D_i^A, \text{ for } i=1, \dots, I \quad (8)$$

$$I_j^{R+} - I_j^{R-} = q_j^R - D_j^R, \text{ for } j=1, \dots, J \quad (9)$$

$$I_i^{A+}, I_i^{A-}, I_j^{R+}, I_j^{R-} \geq 0, \text{ for } i=1, \dots, I, j=1, \dots, J \quad (10)$$

In this reformulation, the first stage problem consists of the objective function (6) subject to constraints (3), (4), and (5). The second stage problem consists of the objective function (7) subject to constraints (2), (8), (9) and (10).  $X = \{q_j^R, j=1, \dots, J\}$  is the vector of first-stage decision variables.  $\omega = \{D_i^A, D_j^R, i=1, \dots, I, j=1, \dots, J\}$  is the vector of random parameters and  $Y = \{I_i^{A+}, I_i^{A-}, I_j^{R+}, I_j^{R-}, i=1, \dots, I, j=1, \dots, J\}$  is the vector of new decision variables defined in the second stage.  $I_i^{A+}$  and  $I_j^{R+}$  denote the overstocking quantities,  $[q_i^A - D_i^A]_+$  and  $[q_j^R - D_j^R]_+$ ; whereas,  $I_i^{A-}$  and  $I_j^{R-}$  denote the under-stocking quantities,  $[D_i^A - q_i^A]_+$  and  $[D_j^R - q_j^R]_+$ . The values of these second-stage decision variables depend on the variation of demand. Note that, for simplicity, the term  $\min \{q_j^R, D_j^R\}$  in the (BP) model is replaced by  $q_j^R - I_j^{R+}$  in BP-2SLPR.

## 4.2 Solution Methodology

We choose the approximate approach that is based on the sampling average approximation (SAA) scheme (e.g. Ruszczyński and Shapiro, 2003, Mak *et al.*, 1999) to solve our model. In this approach, a random sample  $\omega^1, \omega^2, \dots, \omega^N$  of  $N$  realizations (scenarios) of the random vector is generated outside of an optimization procedure and the expectation of second-stage objective function is approximated by the sample average function:

$$E_\xi [g(x, \omega)] = \frac{1}{N} \sum_{i=1}^N g(x, \omega^i). \text{ Therefore, (BP-2SLPR) is approximated by the following SAA}$$

problem (BP-2SLPR-SAA):

$$\begin{aligned} \text{Max } \hat{\pi}_H = & \sum_{i=1}^I (w_i^A \times q_i^A) + \sum_{j=1}^J (p_j^R q_j^R - w_j^R q_j^R) + \frac{1}{N} \sum_{s=1}^N \sum_{i=1}^I (w_i^A I_{is}^{A-}) \\ & + \frac{1}{N} \sum_{s=1}^N \sum_{j=1}^J (-v_j^R I_{js}^{R-} + s_j^R I_{js}^{R+} - p_j^R I_{js}^{R+}) \end{aligned} \quad (11)$$

subject to:

$$\sum_{i=1}^I (I_{is}^{A-}) - \sum_{j=1}^J (q_j^R - I_{js}^{R+}) \leq (L_{UB} - 1) \times l_0 - \sum_{i=1}^I (q_i^A), \text{ for } s = 1, \dots, N \quad (12)$$

$$\sum_{i=1}^I (I_{is}^{A-}) - \sum_{j=1}^J (q_j^R - I_{js}^{R+}) \geq (L_{LB} - 1) \times l_0 - \sum_{i=1}^I (q_i^A), \text{ for } s = 1, \dots, N \quad (13)$$

$$q_j^R \leq Q_j^R, \text{ for } j = 1, \dots, J \quad (3)$$

$$\sum_{j=1}^J (w_j^R \times q_j^R) \leq W^R \quad (4)$$

$$I_{is}^{A+} - I_{is}^{A-} = q_i^A - D_{is}^A, \text{ for } i = 1, \dots, I, s = 1, \dots, N \quad (14)$$

$$I_{js}^{R+} - I_{js}^{R-} = q_j^R - D_{js}^R, \text{ for } j = 1, \dots, J, s = 1, \dots, N \quad (15)$$

$$q_j^R \geq 0, \text{ for } j = 1, \dots, J \quad (5)$$

$$I_{is}^{A+}, I_{is}^{A-}, I_{js}^{R+}, I_{js}^{R-} \geq 0, \text{ for } i = 1, \dots, I, j = 1, \dots, J, s = 1, \dots, N \quad (16)$$

In (BP-2SLPR-SAA), the second stage decision variables,  $I_i^{A+}, I_i^{A-}, I_j^{R+}, I_j^{R-}$ , are redefined as  $I_{is}^{A+}, I_{is}^{A-}, I_{js}^{R+}, I_{js}^{R-}$  for each sample of random demands as the values of these variables depend on the realized scenarios of the random parameters in the model. Therefore, the optimal values of these variables are the sample average of the optimal values of  $I_{is}^{A+}, I_{is}^{A-}, I_{js}^{R+}, I_{js}^{R-}$  obtained under each sample of random demands. In (BP-2SLPR), constraints involving second stage decision variables, (2), (8) and (9) are redefined for each sample of random demands as (12), (13), (14) and (15).

The basic idea of the solving procedure consists of generating an approximate solution, which is the solution of a number of instances, say  $M$ , of the SAA problems, each with  $N$  sampled scenarios. The quality of a candidate solution is then tested by bounding the optimality

gap between the true objective value and the expected objective value through standard statistical procedures. A sampling evaluation procedure based on common random numbers (CRN) is used to construct the confidence interval for the optimality gap. The complete solving procedure is described as follows:

Step 1. Generate i.i.d. batches of samples.

Generate  $M$  independent identical distributed (i.i.d.) samples each of size  $N$ . i.e.

$$\begin{bmatrix} \omega_1^m \\ \dots \\ \omega_n^m \\ \dots \\ \omega_N^m \end{bmatrix} = \begin{bmatrix} D_{1,1}^{Rm} \dots D_{1,j}^{Rm} \dots D_{1,J}^{Rm} & D_{1,1}^{Am} \dots D_{1,i}^{Am} \dots D_{1,I}^{Am} \\ \dots \\ D_{n,1}^{Rm} \dots D_{n,j}^{Rm} \dots D_{n,J}^{Rm} & D_{n,1}^{Am} \dots D_{n,i}^{Am} \dots D_{n,I}^{Am} \\ \dots \\ D_{N,1}^{Rm} \dots D_{N,j}^{Rm} \dots D_{N,J}^{Rm} & D_{N,1}^{Am} \dots D_{N,i}^{Am} \dots D_{N,I}^{Am} \end{bmatrix}, \text{ for } m=1, \dots, M$$

where  $[\omega_1^m \dots \omega_n^m \dots \omega_N^m]^T$  is  $m$ th sample of random demands with sample size of  $N$ . More specifically,  $D_{n,j}^{Rm}$  denotes  $n$ th sample of redemption demand of partner  $R_j$ 's rewards in  $m$ th sample replication.  $D_{n,i}^{Am}$  denote  $n$ th sample of accumulation demands for partners' accumulation options respectively in  $m$ th sample replication.

Step 2. Solve the corresponding SAA problems.

For each sample, solve the corresponding SAA problems, BP-2SLPR-SAA and then let  $\hat{\tau}_N^m$  and  $\hat{x}_N^m$  be the corresponding optimal objective values and optimal solutions for  $m=1, \dots, M$ , respectively.

Step 3. Estimate the optimality gap. It includes three sub-steps.

*Step 3.1* Compute  $\hat{x}_N$  by solving the SAA problems as in Step 2 using a sample with sample size  $N'$  larger than  $N$ . Here, the sample of size  $N'$  is generated independently of the

samples used to obtain  $\hat{x}_N^m$ . Take  $\hat{x}_N$  as the candidate feasible solution  $\bar{x}$  of the true problem ( $\pi_H$ ) and then estimate the optimality gap (i.e., how far this candidate solution is away from true optimal solution) by conducting point estimation and confidence interval of the gap through step 3.2, 3.3 and step 4.

*Step 3.2* Estimate the true objective value  $\pi(\bar{x})$  for all replications of samples with sample size  $N$  as follows:

$$\tilde{\pi}_N^m(\bar{x}) = c^T \bar{x} + \frac{1}{N} \sum_{s=1}^N g(\bar{x}, \omega_s^m) = c^T \hat{x}_N + \frac{1}{N} \sum_{s=1}^N g(\hat{x}_N, \omega_s^m), \text{ for } m = 1, \dots, M$$

This step involves the solutions of  $M$  independent second-stage sub-problems,  $g(\hat{x}_N, \omega_s^m)$ , given  $\hat{x}_N$ .

*Step 3.3* Compute the observations of the optimality gap for the candidate solution  $\bar{x}$  as follows.

$$G_N^m(\bar{x}) = \hat{\pi}_N^m - \tilde{\pi}_N^m(\bar{x}), \text{ for } m = 1, \dots, M$$

where  $\hat{\pi}_N^m$  are generated in Step 2 and  $\tilde{\pi}_N^m(\bar{x})$  are generated in Step 3.2.

#### Step 4. Generate confidence interval (CI) of the optimality gap for the candidate solution.

Compute sample mean and sample variance for the optimality gap:

$$\bar{G}_N^M(\bar{x}) = \frac{1}{M} \sum_{m=1}^M G_N^m(\bar{x}) = \frac{1}{M} \sum_{m=1}^M \hat{\pi}_N^m + \frac{1}{M} \sum_{m=1}^M \tilde{\pi}_N^m(\bar{x}), \text{ and}$$

$$S_G^2(\bar{x}) = \frac{1}{M-1} \sum_{m=1}^M (G_N^m(\bar{x}) - \bar{G}_N^M(\bar{x}))^2$$

It is well known that  $\frac{1}{M} \sum_{m=1}^M \hat{\pi}_N^m$  is an unbiased estimator of  $\hat{\pi}_N$  and  $\hat{\pi}_N$  is an upwards biased estimator of  $\pi^*$  ( $\pi^*$  denotes the optimal value of the true problem) in the case of maximization.

Therefore,  $\frac{1}{M} \sum_{m=1}^M \hat{\pi}_N^m = E[\hat{\pi}_N] \geq \pi^*$  holds (see Mak *et al.*, 1999 for more details), which

indicates that  $\frac{1}{M} \sum_{m=1}^M \hat{\pi}_N^m$  provides a valid statistical upper bound for the optimal value  $\pi^*$  of the

true problem. Meanwhile,  $\frac{1}{M} \sum_{m=1}^M \tilde{\pi}_N^m(\bar{x})$  is an unbiased estimator (i.e., the sample average

estimate) of  $\pi(\bar{x})$ . Since  $\bar{x}$  is a feasible solution to the true problem, we have  $\pi(\bar{x}) \leq \pi^*$ . Thus

$\frac{1}{M} \sum_{m=1}^M \tilde{\pi}_N^m(\bar{x})$  is an estimate of a lower bound on  $\pi^*$ . Therefore,  $\bar{G}_N^M(\bar{x}) \geq 0$ .

Compute the approximate  $100(1-\alpha)$  % CI for the mean of the optimality gap:

$$\left[ 0, \bar{G}_N^M(\bar{x}) + \tilde{\epsilon}_G \right], \text{ where } \tilde{\epsilon}_G = \frac{t_{M-1, \alpha} S_G(\bar{x})}{\sqrt{M}}.$$

## 5. Conclusions

In this paper, we proposed a mathematical model to cope with the LRP host firm's decision on aggregate rewards-supply planning in the presence of multiple commercial partners, who offer different redemption and accumulation options to LRP members. Multiple management concerns are considered in our model such as LRP host value creation, liability control, budget and capacity limitations, and demand uncertainties. We also developed a stochastic programming solution procedure based on sampling average approximation (SAA) scheme.

From academic perspective, this study synthesizes and extends previous theoretical studies on supply chain management in the context of an LRP system. The analytical model integrates key features of an LRP system into the regular SC structure. Meanwhile, in contrast to

the most existing papers on LRPs, this study explores the operational issue of efficiently and effectively planning the supply of rewards.

From practical perspective, the mathematical model we proposed can provide analytical solutions and sensitivity analysis to support LRP host's decision making on rewards-supply planning. Our future work consists of implementing the solution procedure and testing it for real-life size problems. We seek also to expand our model to other contract settings between LRP host firm and LRP partners (e.g. revenue sharing contract, option contract, etc.).

### **List of references**

Berman, B., (2006). Developing An Effective Customer Loyalty Program. *California Management Review*, 49(1), 123-150.

Birge, J. R., & Louveaux, F. V., (1997). *Introduction to Stochastic Programming*. New York, NY: Springer.

Caminal, R. & Matutes, C., (1990). Endogenous Switching Costs in A Duopoly Model. *International Journal of Industrial Organization*, 8, 353-373.

Cachon, G.P., (2003). Supply chain coordination with contracts. In: Steve Graves and Ton de Lok (Eds.), *Handbooks in Operations Research and Management Science: Supply Chain Management*. North-Holland.

Diaby, M., & Nsakanda, A.L., (2008). Coping With Revenue Recognition in the Loyalty Reward Programs Industry: A Stochastic Modeling Approach. *Proceedings of the 2008 American Conference on Applied Mathematics*, Cambridge, Massachusetts, 79-86.

Jain, D., & Singh, S.S., (2002). Customer Lifetime Value Research in Marketing: A Review and Further Directions. *Journal of Interactive Marketing*, 16(2), 34-46.

Kim B., Shi, M., & Srinivasan, K., (2001). Reward Programs and Tacit Collusion. *Marketing Science*, 20(2), 99-120.

Kim B., Shi, M., & Srinivasan, K., (2004). Managing Capacity Through Reward Programs. *Management Science*, 50(4), 503-520.

Labbi A., & Berrospi, C., (2007). Optimizing Marketing Planning and Budgeting Using Markov Decision Processes: An Airline Case Study. *IBM Journal of Research and Development*, 51(3/4), 421-431.

Lewis, M., (2005). Incorporating Strategic Consumer Behaviour into Customer Valuation. *Journal of Marketing*, 69, 230-238.

Liu, Y., (2007). Long-term Impact of Loyalty Programs on Consumer Purchase Behavior & Loyalty. *Journal of Marketing*, 71(4), 19-35.

Mak, W., Morton, D.P., & Wood, R. K., (1999). Monte Carlo Bounding Techniques for Determining Solution Quality in Stochastic Programs. *Operations Research Letters*, 24, 47-56.

Meyer-Waarden, L., (2008). The Influence of Loyalty Programme Membership on Customer Purchase Behaviour. *European Journal of Marketing*, 42(1/2), 87-114.

Nsakanda, A. L., Diaby, M., & Cao, Y., (2010). A Predictive Model of Redemption And Liability In Loyalty Reward Programs Industry. *Proceedings of the 43th Annual Hawaii International Conference on System Science*, Jan.5-8, 2010. Computer Society Press.

Ruszczynski, A., & Shapiro, A., (2003). Monte Carlo Sampling Methods. in *Handbooks in OR & MS Vol. 10*, A. Ruszczyński, A. Shapiro (Eds.), Elsevier Science B.V.

Sharp, B., & Sharp, A., (1997). Loyalty Programs and Their Impact on Repeat-Purchase Loyalty Patterns. *International Journal of Research in Marketing*, 14, 473-486.

Shugan, S. M., (2005). Brand Loyalty Programs: Are They Shames? *Marketing Science*, 24(2), 185-193.

Tsay, A.A., Steven, N., & Narendra, A., (1999). Modeling Supply Chain Contracts: A Review. In: S. Tayur, M. Magazine, R. Ganesan (Eds.), *Quantitative models for supply chain management* (pp.299-336). Boston: Kluwer Academic Publishing.

Wang, K.J., Wang, S.M., & Chen, J.C. (2008). A Resource Portfolio Planning Model Using Sampling-based Stochastic Programming and Genetic Algorithm. *European Journal of Operational Research*, 184, 327-340.

Zhang, Z. J., Krishna, A., & Dhar, S.K., (2000). The Optimal Choice of Promotional Vehicles: Front-Loaded or Rear-Loaded Incentives? *Management Science*, 46 (3), 348–362.