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Pricing and Assortment Decisions for a Manufacturer Selling through Dual-Channels

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Abstract

In many supply chains, the manufacturer sells not only through an independent retailer, but also through its own direct channel. This work studies the pricing and assortment decisions in such a supply chain in the presence of inventory costs. In our model, the retailer offers a subset of the assortment that the manufacturer offers through its direct channel. We model the customer demand by building on the nested-logit model, which captures the customer's choice between the manufacturer and the retailer. This model produces several insights into the optimal pricing strategies of the manufacturer. For example, we find that variants with high demand variability will carry a lower wholesale price. Furthermore, we characterize scenarios in which the manufacturer's and retailer's assortment preferences are in conflict. In particular, the manufacturer may prefer the retailer to carry items with high demand variability while the retailer prefers items with low demand variability.

Keywords: pricing; assortment planning; inventory; supply chain; dual channel.

1. Introduction

In a supply chain setting, the pricing of an assortment is a critical decision not only for the seller itself, but also for its supplier. This pricing question becomes even more critical in supply chains where the manufacturer is both a supplier to and competitor of the retailer. Take the relationship between Sony and Best Buy as an example; specifically the Sony VAIO BZ560 line of laptop computers. SonyStyle.com sells, at the bare minimum, 45 different configurations for the Sony VAIO BZ560 computer. In contrast, Best Buy offers the customer only one VAIO BZ560 configuration. Motivated by such channel relationships, in this work we consider the pricing and assortment selection problems that arise in a supply chain where the manufacturer uses dual channels.

The marketing literature suggests that a store's assortment is almost as important as its price profile and location in driving the store traffic, see, for example, Zhang et al. (2009) and the references therein. Hence, in this work, we model the customer's choice of channel as a function of the assortment and prices offered by both channels. In particular, we use the nested logit model to capture the consumer choice: The customer first chooses the channel she wants to purchase from (if any) and, subsequently, decides which product to purchase from her chosen channel. This demand model allows us to account for the effect of both channels' assortments and prices on the demand observed by each channel.

There are several assumptions one can make regarding who carries inventory in this dual-channel structure and where. In keeping with the motivating example, this work considers a manufacturer (e.g. Sony) who sells a build-to-order product through its direct channel while meeting the orders from the retailer. As for the retailer (e.g. Best Buy), it is assumed that it keeps inventory of the final (assembled) products and meets the observed demand with this inventory. Because the retailer must make stock level decisions before observing the customer demand, there exists the possibility of demand-inventory mismatch at the retailer. Hence, the model accounts for the inventory-related costs associated with the demand-inventory mismatch at the retailer. In the case where the demand at the retailer exceeds the inventory level, the retailer is allowed to procure additional products from the manufacturer. For example, if the demand of a specific VAIO BZ560 computer is greater than the amount Best Buy had in stock, then Best Buy could order additional units from Sony to meet the excess demand.

A strength of this model is that it allows us to analyze the effect of inventory-related costs on the pricing decisions. The selling prices charged by the manufacturer's direct channel and the retailer follow an equal effective margin property similar to that described in Rodríguez and Aydın (2010). In addition, this work characterizes the optimal wholesale prices that the manufacturer charges to the retailer. For example, we find that, everything else being equal, the manufacturer will charge lower wholesale prices for variants with larger demand variability.

This setting where the manufacturer is able to sell through two channels, can be used to study if the manufacturer benefits from selling through two separate channels. In practice one can find both success and failure stories about engaging in dual (or hybrid) sales channel strategies. For example, although Dell has been very successful selling directly to customers, in 2006 they saw their profits and market share decline significantly. The reaction to this decline came in 2007, when Dell successfully embraced a hybrid strategy by adding resellers to their channel mix.[1] In contrast, by 2008, Gateway, another computer business, moved from engaging in dual sales channels to only selling indirectly to customers.[2] Inspired by these examples, in this work we investigate the benefits of engaging in dual-sales channels.

For a build-to-order manufacturer and a retailer engaging in dual sales channels, another relevant question is what assortment to offer through the retailer. More often than not, the retailer offers only a subset of what the manufacturer's direct channel offers as indicated by the Sony VAIO example discussed earlier. Depending on the power structure in the supply chain, the retailer's assortment can be decided by the retailer itself or it could be dictated by the manufacturer. We study both options. Moreover, we study different sequences of decision-making that allow various scenarios regarding the timing of assortment and pricing decisions.

We first explore problems where there is no fixed cost for offering a product variant and there is no capacity limitation on the number of variants to carry. In such cases, we find that if the manufacturer's pricing decisions precede the retailer's assortment selection, both the retailer and the manufacturer will be best off by offering every available product. However, if the retailer's assortment selection precedes the manufacturer's pricing decisions, then the retailer may strategically leave certain variants out of its assortment.

This work also studies cases where there is a fixed cost for carrying a variant or where there is a limit on the size of the assortment. When there is a limit on the size of the retailer's assortment, we find that the manufacturer and the retailer may disagree about which product to sell through the retailer, because the manufacturer prefers products with higher demand variability while the retailer prefers products with lower demand variability. When both the manufacturer and the retailer incur a fixed cost for offering a product through the retailer, we find that the manufacturer's preferred assortment is larger than the retailer's, even when the manufacturer's fixed cost per product is slightly higher.

The rest of the paper is organized as follows. In the following section we review the related literature. Section 3 describes the model. The pricing problem is analyzed in Section 4 and the benefit of adding an indirect channel is analyzed in Section 5. In Section 6 we explore the assortment decisions. Finally, Section 7 summarizes the results and contributions. The

[1] Kellogg Insight: Focus on Research, A new strategy for Dell. Retrieved from `http://insight.kellogg.northwestern.edu/index.php/Kellogg/article/a_new_channel_strategy_for_dell` on 9/29/2009.

[2] Betanews.com, End of an era: Gateway stops selling PCs directly to customers. Retrieved from `http://www.betanews.com/article/End-of-an-era-Gateway-stops-selling-PCs-directly-to-customers/1217271457` on 9/29/2009.

proofs are included in Appendix C.

2. Literature Review

The analysis of distribution systems has received considerable attention in the operations and marketing literature. In the context of the broad literature on distribution systems, the problem studied in this paper belongs to the subset that deals with multiple-channel distribution systems, in which a supplier sells through more than one channel. Cattani et al. (2004) present a recent and extensive literature survey on the coordination of multiple channels.

The multiple-channel distribution system studied in this work belongs to the narrower subset of dual-channel systems, in which the supplier sells through two channels only. The interest in dual channel systems (which have also been labeled as "hybrid distribution") dates back to as early as 1965 (Preston and Schramm, 1965). However, the interest in dual-channel systems has been revived in recent years due to the tradeoffs presented by e-commerce. Agatz et al. (2008), Swaminathan and Tayur (2003) and Tsay and Agrawal (2004) review the literature dealing with multiple-channels that arise in the e-business setting.

One could separate between two streams of work on the dual-channel distribution systems. The first stream of work deals with questions surrounding how much to stock and where to keep that stock in the distribution system, see for example Boyaci (2005), Alptekinoğlu and Tang (2005), Chiang and Monahan (2005), Moinzadeh (2003) and Seifert et al. (2006). Another example is Zhao (2008), which adds the pricing problem to the inventory decision. In commonality with this work, this paper takes into consideration inventory costs of the products offered. However, we simplify the inventory aspect of the problem by assuming that the inventory levels are chosen to satisfy an exogenously fixed service level and the only stock-keeping location is the retailer. These assumptions are in line with our motivating examples, which revolve around build-to-order manufacturers adding a retailer to their channel mix.

The second stream of work in dual-channel systems deals with how the prices should be set and/or coordinated in this distribution system., e.g. Cattani et al. (2006), Chiang and Chhajed (2005), Kumar and Ruan (2006) and Rhee and Park (2000). This work is related to this second stream of research in that we study, among other things, the pricing decisions in a dual-channel system. Earlier work that is particularly related to the type of pricing problems that arise in this work are Chiang et al. (2003) and Tsay and Agrawal (2004), who treat the manufacturer's channel structure as a decision variable, i.e., manufacturer decides whether or not to use a dual-sales channel. They study the effects of the channel structure on the pricing strategies and profits. Tsay and Agrawal (2004) build on Chiang et al. (2003) by incorporating sales effort and the unit cost of supplying an item; however, they restrict the selling prices to be the same in both channels. This work differs from Chiang et al. (2003) and Tsay and Agrawal (2004) in a number of ways. In particular, this work incorporates demand uncertainty and explicitly models the inventory costs associated with demand-stock mismatches at the retailer.

This work also addresses the question of whether it is always beneficial to sell through dual-channels. There has been some work on the question of channel design, especially when considering the distribution costs, e.g. Rangan (1987) and Chiang et al. (2003). In our case, we do not explicitly model distribution costs but we incorporate inventory costs and compare the expected profits for the manufacturer under both scenarios.

What separates this work from all the work that came before it is that we model a dual-channel system in which each channel sells an assortment of substitutable products, and we analyze the assortment decisions.

3. Model Description

Consider a product that can be purchased through two channels: directly from the manufacturer and through an independent retailer. Take a Dell Inspiron desktop computer as an example: A customer can purchase a Dell Inspiron computer directly from Dell.com (the manufacturer's direct channel) where the customer configures the computer by choosing from at least 2 models, 8 colors, 7 processors, 3 operating systems, 5 memory choices, 5 hard drive capacities (at least 8,400 different variants of the Inspiron). On the other hand, the customer may choose to purchase a Dell Inspiron from Best Buy (retailer) by choosing from five different pre-configured Inspiron computers. Notice that Best Buy is offering a subset of the variants that could be purchased from Dell.com. In keeping with this scenario, we model a build-to-order manufacturer, who offers an assortment of all possible variants; we denote this set of variants with S^M. The retailer in our model, on the other hand, offers a subset of the variants in S^M and keeps stock of the variants it carries. Let S^R denote the set of variants carried by the retailer.

In our model, the pricing decisions available to the manufacturer are the prices for the direct channel, p_k^M for variant $k \in S^M$ (i.e. prices charged to the customers who purchase from the manufacturer), and the wholesale price charged to the retailer, w_k for variant $k \in S^R$. On the other hand, the retailer chooses the price it charges to its customers. Let p_k^R denote the retailer's selling price for variant $k \in S^R$.

3.1 Customer Demand Model

Consider an individual customer. We model the individual customer's decision building on the nested-logit model. In the nested-logit model the customer choice is modeled as a sequential process, where the customer first chooses one of many 'nests' of items and, conditional on the choice of the nest, the customer chooses what specific item to purchase from the nest. The choices in each stage follow a multinomial logit choice model (MNL). The nested logit model leads to a closed form expression for the probability that a customer purchases a specific item in a given nest. For more on the nested-logit model see Anderson and de Palma (1992). In our model, the nests are the manufacturer's assortment, the retailer's assortment and an external alternative. If the customer decides to purchase from the manufacturer or the retailer, then she decides on the specific product to purchase. For the sake of exposition, we first describe the customer's variant choice given that the customer already decided what channel to purchase from (the retailer or the manufacturer's direct channel).

3.1.1 Deciding what variant to purchase

In this stage, the customer chooses which variant to purchase from the assortment offered by the channel she chose in the first stage. Consider an individual customer who decided to purchase from channel $n \in \{M, R\}$, where $n = M$ refers to the manufacturer's direct channel and $n = R$ refers to the retailer. Let $\boldsymbol{p}^n$ denote the vector of prices for the variants offered by channel n.[3] In keeping with the nested logit choice model, we model the customer's choice of variant using the multinomial logit (MNL) choice model. (For details on the MNL model, see, for example, Ben-Akiva and Lerman, 1985.) Let U_k^n denote the customer's utility from the variant $k \in S^n$. Following the MNL model, U_k^n is given by

$$U_k^n = \alpha_k - p_k^n + \xi_k^n, \text{ for } k \in S^n, n \in \{M, R\},$$

where α_k is the customer's expected utility from variant $k \in S^n$, p_k^n is the price of variant k in channel n and ξ_k^n is a random error term with a Gumbel distribution with mean zero and scale parameter $\mu_2 > 0$. The scale parameter μ_2 can be interpreted as the degree of heterogeneity across the variants offered by a given channel; when μ_2 is close to zero the variants become

[3] As a notational convention; we use bold symbols to denote vectors, e.g. $\boldsymbol{p}^n$ is the vector of prices charged by channel n to its customers.

perfect substitutes. In this setting, the customer chooses the variant that maximizes his utility and the probability that the customer will choose variant k is

$$q_k^n(S^n, \boldsymbol{p}^n) = \frac{v_k(p_k^n)}{\sum_{k \in S^n} v_k(p_k^n)} \text{ for all } k \in S^n, \tag{1}$$

where

$$v_k(p_k^n) := \exp([\alpha_k - p_k^n]/\mu_2) \text{ for } k \in S^n. \tag{2}$$

3.1.2 Channel choice

Let U^M and U^R denote the customer's random utility of purchasing from the manufacturer and from the retailer, respectively. The nested logit model posits that the utility of purchasing from a nest is the expected utility of the utility-maximizing choice in that nest plus a Gumbel error term. Hence, following the nested logit model, $U^n := E\left[\max_k U_k^n\right] + \xi_n$ for $n = R, M$ where ξ_n is a Gumbel random term with mean zero and scale parameter μ_1, where $\mu_1 > 0$. The parameter μ_1, in contrast to μ_2, represents the heterogeneity of assortments across the two channels. Hence, we expect that $\mu_1 > \mu_2$. Given that U_k^n's are Gumbel random variables (refer to Section 3.1.1) and the Gumbel distribution is closed under maximization, $\max_k U_k^n$ is again Gumbel, and its expected value is,

$$E\left[U^n\right] = E\left[\max_k U_k^n\right] = \mu_2 \ln\left[\sum_{k \in S^n} v_k(p_k^n)\right]. \tag{3}$$

Similarly, let U^0 denote the random utility of an external alternative. Here, U^0 can be thought of as the aggregate utility from a number of choices exogenous to our model, such as the utility from purchasing the product from an alternate firm or the utility from purchasing a different product altogether. This utility of the external alternative, U^0, is itself Gumbel with a mean of $\alpha_0\mu_1$.

The nested logit model then yields the following expression for the probability that the customer purchases from channel n, denoted by $\tau^n(S^M, S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)$:

$$\tau^n(S^M, S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) = \frac{\exp(E[U^n]/\mu_1)}{\exp(E\left[U^0\right]/\mu_1) + \exp(E\left[U^R\right]/\mu_1) + \exp(E[U^M]/\mu_1)}$$

for $n \in \{M, R\}$. Using (3) to substitute for $E[U^n]$ in the above equation, we obtain:

$$\tau^n(S^M, S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) = \frac{\left[\sum_{k \in S^n} v_k(p_k^n)\right]^{\mu_2/\mu_1}}{\exp\left(\alpha_0\right) + \left[\sum_{k \in S^R} v_k(p_k^R)\right]^{\mu_2/\mu_1} + \left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}. \tag{4}$$

3.1.3 The Aggregate Demand

According to the model we have described so far, an individual customer purchases from the channel $n \in \{M, R\}$ with probability $\tau^n(S^M, S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)$ and chooses variant $k \in S^n$ with probability $q_k^n(S^n, \boldsymbol{p}^n)$. Thus, the probability that an individual purchases variant k from channel n is

$$\tau^n(S^M, S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) q_k^n(S^n, \boldsymbol{p}^n).$$

We use the probability above as the starting point to model the aggregate demand for a product. Specifically, we assume that the aggregate demand observed by channel n for product k, denoted by $D_k^n(S^M, S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)$, is

$$D_k^n(S^M, S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) \sim \tau^n(S^M, S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) q_k^n(S^n, \boldsymbol{p}^n)\epsilon_k \text{ for } k \in S^n,$$

where ϵ_k's are i.i.d normal random variables with mean one and standard deviation σ_k. Notice that σ_k amounts to the coefficient of variation of customer's demand for variant $k \in S^n$.

3.2 The Firms' Profit Functions

We treat the manufacturer's assortment as fixed and hence we drop S^M from the argument lists of the functions. On the other hand, we treat the retailer's assortment, S^R, as a decision variable. This modeling choice is aligned with our motivating example, in which the manufacturer offers all possible variants while the retailer offers only a subset of them. We consider both the case where the retailer chooses its own assortment, and the case where the manufacturer decides what to offer through the retailer.

We assume that the manufacturer has two pricing decisions to make: the direct channel prices, p_k^M, and the wholesale price for each variant, denoted by w_k for $k \in S^R$. The retailer, on the other hand, needs to determine its own selling prices, p_k^R for variant $k \in S^R$. We consider several scenarios regarding the sequencing of decisions, including the pricing decisions and the assortment decision. For the sake of exposition, we next describe the retailer's and manufacturer's profit functions given the prices and assortments and delay an explanation of the sequence of events until the next section.

3.2.1 Retailer's Expected Profit

We consider a one-period problem. Recall that we model a make-to-stock retailer who procures the finished product from the manufacturer and faces random demand. Hence, inventory decisions at the retailer are taken into account and we assume these decisions are made after all pricing and assortment decisions have been made by both the manufacturer and retailer.

Let y_k denote the retailer's stock level of variant $k \in S^R$. We assume that the retailer will stock y_k units at the beginning of the period to satisfy a type-1 service level objective, which is uniquely determined by the exogenously fixed parameter, z_k, where

$$z_k := \frac{y_k - \mu_k^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})}{\widehat{\sigma}_k^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})}, \tag{5}$$

and $\mu_k^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ and $\widehat{\sigma}_k^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ denote the mean and standard deviation of the demand for variant k observed by the retailer.

At the beginning of the period, the retailer places the orders, y_k's, with the manufacturer. Next, the demand at the retailer, $D_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ for $k \in S^R$, is realized. For each variant, two outcomes may arise in this setting for the retailer: either the variant's stock is sufficient to meet the demand or not. In the former case, the retailer incurs a cost co_k for each unit of leftover inventory. In the latter case, we assume that the retailer places an additional order with the manufacturer to meet the excess demand, at a unit cost cu_k, which is in addition to the wholesale price. The cost cu_k may be interpreted as the additional cost associated with expediting a shipment in order to meet the excess demand. In this setting, the retailer's expected profit is

$$\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R}) = \sum_{k \in S^R} \left[(p_k^{R} - w_k) E\left[D_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) \right] - L_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) \right], \tag{6}$$

where

$$L_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) = cu_k E\big[D_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) - y_k\big]^+ + co_k E\big[y_k - D_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})\big]^+. \tag{7}$$

Here, the function $L_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ represents the expected overage and underage cost associated with variant $k \in S^R$. The profit function in (6) can be simplified to the following:

$$\begin{aligned} &\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R}) = \\ &\tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) \sum_{k \in S^R} \begin{bmatrix} (p_k^{R} - w_k) q_k^{R}(S^R, \boldsymbol{p}^{R}) \\ - \sigma_k \big[co_k z_k + (co_k + cu_k) I_N(z_k)\big] q_k^{R}(S^R, \boldsymbol{p}^{R}) \end{bmatrix}. \end{aligned}$$

Let us define $\gamma_k := \sigma_k\big[co_k z_k + (co_k + cu_k) I_N(z_k)\big]$. Notice that γ_k works as a unit cost incurred by the retailer and it depends only on unit overage cost, co_k, unit underage cost, cu_k, the demand's coefficient of variation, σ_k, and the service level z_k. Using this definition, we can now write the retailer's expected profit function as follows:

$$\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R}) = \tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) \sum_{k \in S^R} (p_k^{R} - w_k - \gamma_k) q_k^{R}(S^R, \boldsymbol{p}^{R}). \tag{8}$$

3.2.2 Manufacturer's Expected Profit

The manufacturer builds to order and faces two sources of demand: one from the direct channel and the other from the retailer. The sequence of events for the manufacturer is the following: At the beginning of the period, the manufacturer receives an order from the retailer and meets that order. Throughout the period, the manufacturer continues to build to order to meet the demand from its direct channel. At the end of the period, the retailer may observe excess demand over the initial stocking quantity. When that happens, the retailer will backorder excess demand and will procure the needed quantity from the manufacturer. The manufacturer will meet the retailer's additional demand charging the same wholesale price as before.

Given that the retailer backorders the excess demand from the manufacturer, the quantity that the manufacturer sells to the retailer is the maximum of the retailer's initial order quantity, y_k, and the demand observed by the retailer, $D_k^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)$. The quantity that the manufacturer sells through the direct channel, on the other hand, is simply the demand from the direct channel, $D_k^M(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)$. Let c_k denote the unit production cost of variant k. The manufacturer's expected profit is

$$\Pi^M(S^R, \boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R) = \sum_{k \in S^M} (p_k^M - c_k) E\big[D_k^M(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)\big] + \sum_{k \in S^R} (w_k - c_k) \left(y_k + E\left[D_k^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) - y_k\right]^+ \right). \quad (9)$$

Notice from (9) that the first term corresponds to the expected profit from the direct channel and the second term corresponds to the profit collected by selling to the retailer. We substitute for y_k in (9) using the definition in (5). Note that, by the assumption of normal demand, we can replace $E\left[D_k^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) - y_k\right]^+$ with $\widehat{\sigma}_k^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) I_N(z_k)$. Substituting $\mu_k^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) = \tau^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) q_k^R(S^R, \boldsymbol{p}^R)$, $\widehat{\sigma}_k(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) = \tau^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) q_k^R(S^R, \boldsymbol{p}^R) \sigma_k$, and

$$E\big[D_k^M(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)\big] = \tau^M(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) q_k^M(\boldsymbol{p}^M),$$

we obtain that the profit for the manufacturer is given by

$$\Pi^M(S^R, \boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R) = \tau^M(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) \sum_{k \in S^M} (p_k^M - c_k) q_k^M(\boldsymbol{p}^M) + \tau^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R) \sum_{k \in S^R} (w_k - c_k) \theta_k q_k^R(S^R, \boldsymbol{p}^R), \quad (10)$$

where

$$\theta_k := \sigma_k \left(z_k + I_N(z_k)\right) + 1. \quad (11)$$

We refer to θ_k as the "safety stock factor" for variant k. Notice the significance of the safety stock factor: For each variant, the expected quantity sold by the manufacturer to the retailer is the variant's expected demand times its safety stock factor, which itself depends on the service level and demand's coefficient of variation, as shown in (11). This factor is always larger than one. It captures the fact that the total quantity that the retailer buys from the manufacturer is an amplification of the expected demand. To understand the intuition behind this amplification, recall that the retailer backorders from the manufacturer whenever there is a shortage. Therefore, the total quantity the retailer buys from the manufacturer is never below the demand, but it may exceed the demand when the retailer overstocks. The factor θ_k captures this effect.

4. The Pricing Problem in the Dual Channel

In this section we explore the retailer's and manufacturer's pricing problem assuming a Stackelberg game where the manufacturer is the leader. To examine the retailer's pricing decision, we assume that the retailer's assortment, S^R, is exogenously fixed. The sequence of decisions for this game is the following: (1) the manufacturer picks the wholesale prices, w_k, and direct channel prices, p_k^{M}, (2) the retailer sets its prices, p_k^{R}. We first analyze the retailer's pricing decision in response to the manufacturer's wholesale prices, $\boldsymbol{w}$, and direct channel prices, $\boldsymbol{p}^{M}$. As stated in the following proposition, we find that the retailer will price its products following an equal effective margin property.

Proposition 1. *Consider variant k in the retailer's assortment S^R. Let $m_k^{R} := p_k^{R} - w_k - \gamma_k$ denote the effective profit margin of variant k. Given the vector of wholesale prices, $\boldsymbol{w}$, the vector of direct channel prices, $\boldsymbol{p}^{M}$, and the retailer's assortment, S^R, any price vector that is optimal for the retailer is such that all variants have the same effective profit margin, i.e., $m_k^{R} = m^{R}$ for all $k \in S^R$.*

Proposition 1 reduces the retailer's pricing decision to picking a single effective margin, m^{R}, for all the variants in its assortment S^R. This result shows the same pricing structure as a result in Rodríguez and Aydın (2010).

Recall the expression for the retailer's profit, $\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R})$, given by (8). Using the result of Proposition 1, we obtain:

$$\sum_{k \in S^R} (p_k^{R} - w_k - \gamma_k) q_k^{R}(S^R, \boldsymbol{p}^{R}) = m^{R} \sum_{k \in S^R} q_k^{R}(S^R, \boldsymbol{p}^{R}) = m^{R}.$$

Using the equality above, we can rewrite the retailer's profit, $\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$, as

$$\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) = \tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) m^{R}. \tag{12}$$

Furthermore, the probability that the customer chooses the retailer, $\tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$, can be rewritten as a function of m^{R} instead of the price vector $\boldsymbol{p}^{R}$. To this end, for variant k in retailer's assortment S^R, define the retailer's surplus associated with variant k as

$$\eta_k^{R} := \alpha_k - w_k - \gamma_k \text{ for } k \in S^R. \tag{13}$$

Note that the surplus is the customer's expected utility from the variant minus the costs the retailer incurs for carrying that variant. Using the above definition, we can define the function $v_k^{R}(m^{R})$ as follows:

$$v_k^{R}(m^{R}) := \exp([\eta_k^{R} - m^{R}]/\mu_2) \text{ for } k \in S^R. \tag{14}$$

Using (13) and (14), we can now write $\tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ in (4) as a function of the retailer's effective margin, m^{R}, instead of the retailer's price vector, $\boldsymbol{p}^{R}$,

$$\tau^{R}(S^R, \boldsymbol{p}^{M}, m^{R}) = \frac{\left[\sum_{k\in S^R} v_k^{R}(m^{R})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^R} v_k^{R}(m^{R})\right]^{\mu_2/\mu_1}}. \tag{15}$$

Using the expression above, we rewrite the retailer's expected profit, $\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ in (12), as a function of the effective margin, m^{R} instead of the price vector, $\boldsymbol{p}^{R}$

$$\Pi^R(S^R, \boldsymbol{p}^{M}, m^{R}) = \tau^{R}(S^R, \boldsymbol{p}^{M}, m^{R}) m^{R}. \tag{16}$$

The following proposition uses the redefined profit function in (16) to characterize the retailer's optimal margin as a function of the wholesale prices, $\boldsymbol{w}$, direct channel prices, $\boldsymbol{p}^{M}$, and the retailer's assortment, S^R.

Proposition 2. *Given the vector of wholesale prices, $\boldsymbol{w}$, the vector of direct channel prices, $\boldsymbol{p}^{M}$, and the retailer's assortment, S^R, the retailer's optimal effective margin is the unique value of $m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})$ that satisfies*

$$m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}) = \frac{\mu_1}{1 - \tau^{R}(S^R, \boldsymbol{p}^{M}, m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}))}.$$

Proposition 2 characterizes the retailer's optimal pricing response to any vector of wholesale prices, $\boldsymbol{w}$, and direct channel prices, $\boldsymbol{p}^{M}$, chosen by the manufacturer. Given the retailer's optimal pricing response, we next analyze the manufacturer's optimal pricing decisions.

In the manufacturer's profit function, the retailer's price vector can now be replaced with the retailer's effective margin. The details of this simplification are provided in Appendix C. This simplification facilitates further analysis of the manufacturer's pricing problem. The next proposition describes the properties of the optimal direct channel and wholesale prices.

Proposition 3. *Define $m_k^{M} := p_k^{M} - c_k$ as the manufacturer's effective profit margin for $k \in S^M$ sold through the direct channel and $\overline{w}_k := (w_k - c_k - \mu_2)\theta_k$ as the weighted wholesale price for variant $k \in S^R$. At any optimal vector of wholesale prices, $\boldsymbol{w}$, and optimal vector of direct channel prices, $\boldsymbol{p}^{M}$:*

(a) All variants $k \in S^M$ will have the same effective margin, i.e. $m_k^{M} = m^{M}$ for all $k \in S^M$.

(b) All variants $k \in S^R$ will have the same weighted wholesale price, i.e. $\overline{w}_k = \overline{w}$ for all $k \in S^R$.

The result in Proposition 3 (a) does not comes as a surprise given that we found the same structure in different settings, where the effective margin (i.e. gross margin net of unit inventory cost) must be the same for all variants in the channel. In this case, what is the same across variants is the gross margin because the manufacturer's direct channel has no inventory-related costs.

Proposition 3 (b) has important consequences regarding the effect of demand variability on the wholesale prices, as we state in the following proposition.

Proposition 4. *Suppose that all variants are the same in all respects but θ_k, i.e. $\alpha_k = \alpha$, $c_k = c$ and $\gamma_k = \gamma$ for all $k \in S^R$. Let i and j be two of the variants carried in the retailer's assortment. If $\theta_i < \theta_j$ then $w_i > w_j$. Furthermore, if i and j are identical in all respects but the demand's coefficient of variation, with $\sigma_i < \sigma_j$, then $w_i > w_j$.*

Proposition 4 states that the larger the demand variability of a variant (measured by demand's coefficient of variation), the lower the variant's wholesale price. In general, the proposition indicates that variants with higher safety stock factors will prompt lower prices from the manufacturer. The intuition behind this result lies in the safety stock factor, θ_k. Suppose we start with a number of products identical in all respects, and we then increase the coefficient of variation for one of them, say variant h. Now, the safety stock factor for variant h also increases. In other words, for variant h, the retailer's order quantity is going to be a larger amplification of the expected demand compared to other variants. Hence, this variant now becomes more attractive for the manufacturer, so the manufacturer decreases the wholesale price to increase the market share of this variant among others.

Using Proposition 3 we can recast the manufacturer's profit as a function of a single effective wholesale price (instead of a wholesale price vector) and a single effective margin for the direct channel (instead of a price vector for the direct channel). These simplifications, which are presented in detail in the appendix, allow us to further analyze assortment selection and pricing in a dual-channel setting.

5. Effects of Adding a Retailer to a Direct Channel on the Manufacturer's Profit

In this section, we explore how adding an indirect channel (i.e. retailer) influences a manufacturer currently selling directly to the customer. In order to understand the benefits of dual-sales channels, we first explore the effect of adding an indirect channel on the pricing decisions.

Recall that the manufacturer decides the wholesale prices and direct-channel prices. We find that the manufacturer will always increase the gross margin used by its direct channel when it adds an indirect channel.

Proposition 5. *Let m^{M}_{direct} be the manufacturer's optimal direct channel margin when selling through the direct channel only and m^{M}_{dual} be the manufacturer's optimal direct channel margin when selling through dual channels. Then, $m^{M}_{dual} > m^{M}_{direct}$.*

It is interesting that the manufacturer increases its direct channel margin despite introducing a competing source of product, the retailer, into the channel mix. To understand the intuition behind this result, note that the manufacturer's choice of direct margin is ultimately a trade-off between its sales volume and unit margin. Once the retailer is introduced to the mix, it is true that the volume sold through the direct channel will decrease as some customers will switch to the retailer. However, the manufacturer's total sales volume, combined across both channels, will now be higher than it was when the manufacturer had only the direct channel, because fewer customers will now choose to go with the external alternative. Given that the manufacturer's total sales volume will increase after the addition of the retailer, the manufacturer will respond by increasing its direct channel margin to take advantage of the larger sales volume.

S^R	m^M	$\overline{w}$	m^R	Π^M	Π^R
$\{\emptyset\}$	3.0992	—	—	1.5992	0
$\{1\}$	3.2839	3.1954	1.6586	1.8041	0.1586
$\{2\}$	3.2489	3.1163	1.6278	1.7620	0.1278

Table 1: Manufacturer's profit improvements of adding an indirect-channel to a direct-channel. In this example, $\mu_1 = 1.5$, $\mu_2 = 0.5$, $\alpha_0 = 10$, $\alpha_1 = \alpha_2 = 5$, $c_1 = c_2 = 1$, $\gamma_1 = 0.4034$, $\gamma_2 = 0.08068$ and $\theta_1 = \theta_2 = 1.2008$.

Given that adding a retailer results in an increase in both the total sales volume and the direct channel's effective margin, the manufacturer will always benefit from using a dual-channel strategy.

Proposition 6. *Let Π^M_{direct} be the manufacturer's optimal expected profit when selling through the direct channel only and Π^M_{dual} be the manufacturer's optimal expected profit when selling through dual channels. Then, for any given assortments S^M and S^R we have that $\Pi^M_{dual} \geq \Pi^M_{direct}$.*

While the manufacturer always benefits from adding a retailer to the channel mix, the size of the benefit does depend on the assortment offered through the retailer. In Table 1, we present an example with three alternatives: The first example is one where the manufacturer is only selling directly to the customer, i.e. $S^R = \{\emptyset\}$, in the second example the manufacturer is selling variant 1 through the retailer, i.e. $S^R = \{1\}$, and in the third example the manufacturer is selling variant 2 through the retailer, i.e. $S^R = \{2\}$. Notice that the manufacturer's expected profit, Π^M, increases as the indirect-channel is added as predicted by Proposition 6. However, notice that carrying variant 1 through the retailer is more profitable than carrying variant 2. Also notice that variants 1 and 2 are identical in all respects except for the difference between γ_1 and γ_2, the parameters that capture the inventory-related unit costs associated with variants 1 and 2, respectively. This example highlights two aspects of the problem we are studying: First, the assortment decision on what to offer through the retailer plays an important role on the manufacturer's profit. Second, even when the manufacturer builds to order (as is the case in our model), the inventory costs that the retailer incurs for its assortment will influence not only the retailer's profit, but also the manufacturer's profit. Hence, we next explore the assortment decision on what to offer through the retailer.

6. Assortment Results

In this section, we explore the retailer's assortment decision. Depending on the power structure in the supply chain, the retailer may choose its own assortment or the manufacturer may decide on the assortment it is going to offer through the retailer. Hence, in this section we explore both cases. In terms of the timing of the assortment decision, we consider three alternative scenarios. In Scenario 1, the manufacturer picks the assortment while it is choosing its direct channel and wholesale prices. In Scenario 2, the retailer picks the assortment after the manufacturer makes its pricing decisions. In Scenario 3, the retailer picks the assortment before the manufacturer makes its pricing decisions.

Next we explore Scenario 1. Suppose that the manufacturer can pick the assortment offered through the retailer, S^R. Then the manufacturer will offer every available variant as stated in the next proposition.

Proposition 7. *In Scenario 1, where the manufacturer is choosing the retailer's assortment in addition to setting the direct channel and wholesale prices, the manufacturer will always choose to offer all variants through the retailer.*

This result may seem surprising given the fact that the retailer's assortment is going to compete with the direct channel's assortment. However, the manufacturer can always utilize its ability to manipulate the retailer's demand through its wholesale price choice. Hence, the manufacturer can always price the products to get a favorable combination of market share and margin.

Now consider Scenario 2, where the retailer picks its assortment after the manufacturer makes its pricing decisions. In this scenario, we find that the retailer will always choose to carry all available variants.

Proposition 8. *In Scenario 2, where the retailer picks its assortment after the manufacturer makes its pricing decision, the retailer will choose to offer every available variant.*

The driver of this result in Proposition 8 is the fact that the retailer uses its pricing decision to balance the gross profit and inventory-related costs of its variants so that all variants are equally attractive to be carried in the assortment.

The common ground that joins scenarios 1 and 2 is that they both assume that the assortment decision is made between the manufacturer's pricing decisions and the retailer's pricing decision. In contrast, Scenario 3 assumes that the retailer picks its assortment before the pricing decisions for either firm are made.

	Profit		Prices					
S^R	*Manufacturer*	*Retailer*	$\boldsymbol{p}_1^M$	$\boldsymbol{p}_2^M$	w_1	w_2	p_1^R	p_2^R
$\{1\}$	1.9738	0.1731	4.6535	4.6535	4.5827	–	7.0810	–
$\{2\}$	1.9347	0.1175	4.6231	4.6231	–	4.1709	–	7.7250
$\{1, 2\}$	1.9812	0.1722	4.6603	4.6603	4.6535	4.0084	7.1509	7.6172

Table 2: Manufacturer and Retailer's Profit and Margins for a Two-Variant Assortment. In this example, the variant 1 and 2 are the same in all respects but the demand's coefficient of variation σ_k, where, $\sigma_1 = 0.09$, $\sigma_2 = 0.25$, $\mu_1 = 1.7$, $\mu_2 = 0.3$, $\alpha_0 = 10$, $\alpha_1 = \alpha_2 = 5$, $c_1 = c_2 = 1$, $co_1 = co_2 = 4$, $cu_1 = cu_2 = 4$ and $z_1 = z_2 = 1.7$.

Under Scenario 3, we find that even if the retailer has no constraints on the number of variants to carry nor has a fixed cost for carrying a variant, the retailer may still choose not to carry all possible variants in its assortment. We illustrate this result with an example in Table 2. In this example, the manufacturer has an assortment of two variants and the retailer can choose to carry only one variant, two variants or neither. Observe that when the retailer carries variant 2 in its assortment (either on its own or along with variant 1) the retailer charges larger prices to the customer. The reason for increase in prices is that variant 2 shows higher demand variability and therefore, the inventory related costs associated with carrying this variant are large for the retailer. It is interesting that this will happen even when the manufacturer will choose a lower wholesale price of variant 2 compared to variant 1, as discussed in Proposition 4. In this example, the retailer would choose to leave variant 2 out of its assortment in order to push the inventory related costs down.

6.1 Assortment Selection Problem

An important assumption made in the previous section for Propositions 7 and 8 is that there are no constraints on the number of variants that the retailer can offer and there are no fixed costs for carrying a variant. In this section we relax this assumption for two simplified versions of the problem. First, we explore a problem where the retailer is constrained to offer only one variant. This setting allows us to explore what product characteristics present conflicts between the retailer and the manufacturer when it comes to the assortment offered by the retailer. Second, we explore the case where all variants are identical and there is a fixed cost associated with the number of variants carried at the retailer. This setting allows us to explore conflicts between the manufacturer and the retailer in terms of the size of the retailer's assortment.

6.1.1 Variant Preference

Consider the case where the retailer can offer a single variant. The next proposition compares Scenarios 1 and 2.

Proposition 9. *Suppose that the retailer can offers only one variant from S^M. Then the variant that would be offered through the retailer under Scenario 1 is the same as the variant that would be offered under Scenario 2.*

Recalling the definitions of Scenarios 1 and 2, Proposition 9 states that as long as the retailer's assortment is chosen after the manufacturer sets its prices but before the retailer does, the manufacturer and retailer are in agreement about what variant the retailer should offer.

We next analyze the case where the retailer chooses its assortment before the manufacturer sets its prices: Do the manufacturer and the retailer still agree on what variant to offer through the retailer? In other words, we compare Scenario 1 (where the manufacturer chooses what to offer through the retailer) and Scenario 3 (where the retailer chooses what to offer, but before the manufacturer sets its prices). Proposition 10 identifies certain conditions under which the same variant is chosen under both Scenarios 1 and 3.

Proposition 10. *Consider a manufacturer's assortment that consists of only variants 1 and 2, i.e. $S^M = \{1, 2\}$. Let variants 1 and 2 be the same in all respects but the overage and/or underage costs, i.e. $\alpha_1 = \alpha_2$, $c_1 = c_2$, $\theta_1 = \theta_2$. Under both Scenario 1 and Scenario 3, the variant that will be offered through the retailer is the one with the lower underage/overage cost.*

Proposition 10 presents a setting in which the retailer and the manufacturer will not face a conflict in terms of what should be carried through the retailer. However, the same is not true for variants that differ only in the demand's coefficient of variation. In such a case, we find that there exist situations where the manufacturer, in Scenario 1, will prefer to offer through the retailer the variant with the higher demand's coefficient of variation whereas the retailer, in Scenario 3, will choose the variant that has the lower coefficient of variation. In Table 3, we provide an example showing such a conflict. In this example there are two variants, labeled 1 and 2. They are the same in all respects but the demand's coefficient of variation, σ_k. The retailer decides according to Scenario 3. Here the coefficient of variation for variant 2 is significantly larger than the one for variant 1. Observe that the retailer's expected profit

S^R	Π^M	Π^R	p_1^M	p_2^M	w_1	w_2	p_1^R	p_2^R
{ 1}	5.3526	0.2864	7.9969	7.9969	7.6224	–	10.2340	–
{ 2}	5.3889	0.2597	8.0322	8.0322	–	6.7291	–	10.4254

Table 3: Profits of a Retailer's Single-Variant Assortment. In this example, the variant 1 and 2 are the same in all respects but the demand's coefficient of variation σ_k, where $\mu_1 = 1.7$, $\mu_2 = 0.3$, $\sigma_1 = 0.09$, $\sigma_2 = 0.25$, $\alpha_0 = 10$, $\alpha_1 = \alpha_2 = 10$, $c_1 = c_2 = 1$, $co_1 = co_2 = 4$, $cu_1 = cu_2 = 4$ and $z_1 = z_2 = 1.7$; $\theta_1 = 1.1546$, $\theta_2 = 1.4296$, $\gamma_1 = 0.6252$, and $\gamma_2 = 1.7366$.

is higher when it carries only variant 1, whereas the manufacturer's expected profit is higher when the retailer carries variant 2.

The most significant driver of this conflict is the fact that variant 2 has a higher safety factor, i.e. $\theta_2 > \theta_1$. Hence the manufacturer will show preference for variant 2. However, variant 2 also has higher inventory costs, i.e. $\gamma_2 > \gamma_1$ which makes it less attractive for the retailer. Hence, the retailer prefers variant 1, but the manufacturer prefers variant 2.

6.1.2 Assortment Size

In this section we explore the assortment size preference of the manufacturer and retailer for a simplified version of the base model. Suppose that both the retailer and the manufacturer incur a fixed cost for every product carried in the retailer's assortment. Furthermore, suppose that all variants are identical, i.e., α_k, c_k, γ_k and θ_k are all the same for all $k \in S^M$. We assume that the assortment decision is made after the manufacturer picks its prices, i.e. Scenarios 1 and 2. In this setting, we compare the retailer's and manufacturer's preference for the size of the assortment offered through the retailer. We find that the manufacturer prefers a larger assortment than the retailer.

Proposition 11. *Suppose $\theta_k < 2$. The optimal size of the retailer's assortment is larger under Scenario 1 than under Scenario 2.*

Proposition 11 implies that, when the assortment decision is made after the manufacturer's pricing decision but before the retailer's pricing decision, the manufacturer will always prefer a broader assortment than the one preferred by the retailer. There are two reasons for this discrepancy. First, the retailer incurs inventory costs for each variant in its assortment while the manufacturer doesn't. Second, when the demand falls short of stock level, the retailer sells only up to demand, but the manufacturer's sales quantity is what the retailer stocked. Thus, the manufacturer always sells at least as much as the retailer, and sometimes strictly more, thereby benefiting more from each variant than the retailer does.

7. Conclusion

This work studies a supply chain structure where the upper echelon supplies and competes with the lower echelon. We study this setting by modeling a build-to-order manufacturer selling directly to the customer and also through a retailer. The customer is assumed to be able to purchase from either firm and the manufacturer will offer all available variants, whereas the retailer will offer only a subset of variants.

There are three main challenges that this type of problem poses: First we have the manufacturer's decision to set direct channel prices and wholesale prices, while taking into account the competition from the retailer. Second, the retailer's manages not only prices but also inventory levels. Third, both the retailer and the manufacturer must be concerned about the effects of the retailer's assortment on their profits.

In this setting, we first look at the firms' pricing problems. We find that for the retailer, the pricing structure will be the same as the one found in earlier work (Rodríguez and Aydın, 2010), where the prices should follow an equal effective margin. The optimal wholesale prices on the other hand show a different structure: The wholesale price net of unit production and inventory costs, weighted by the item's safety stock factor, must be the same across all items. Here, the safety stock factor is a function of the item's service level and demand variability. This structure has some immediate implications. For example, if all items are the same but some show higher demand variability, then those items with higher demand variability will have a lower wholesale price attached to them. The rationale behind this result is that, for these items, the retailer's order quantity will represent a larger amplification of the expected demand, compared to other items with lower demand variability. Hence, the manufacturer will want these items to have a larger market share, which can be induced by keeping the wholesale price low.

Given the increasing diversification of channels, we also explore if a manufacturer, who is currently selling through a direct channel, benefits from adding a retailer to its channel mix. This decision is non-trivial because the two channels will be competing against each other. We find that even though the direct channel will observe a decrease in its market share, the manufacturer will increase its total market share (where total market share includes the demand from both channels). Hence, the manufacturer will always benefit from adding a channel, giving it another venue to capture more customers.

One of the main contributions of this work is that is explores the retailer's assortment decision. First, we assume that there are no limitations on the number of variants to offer through the retailer and that there are no fixed costs of carrying a variant. In this setting, we explore scenarios that differ in mainly two aspects: who decides the retailer's assortment (the retailer or the manufacturer itself) and when the assortment decision is made relative to pricing decisions. We find that depending on the sequence of decisions, the manufacturer's preference may conflict with the retailer's. In particular, in certain cases, the manufacturer wishes the retailer to offer items with high demand variability while the retailer opts for low demand variability.

In addition, when both the retailer and the manufacturer incur a fixed cost for the variants carried by the retailer, we find that the manufacturer prefers a broader assortment than the retailer even when the manufacturer's fixed cost is slightly higher than the retailer.

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Appendix

Proofs for Section 4

In preparation for the proofs that follow, we note below the derivatives of purchase probabilities defined in Section 3.1. The following derivatives can be verified through some algebra using the definitions of purchase probabilities in (1) and (4):

$$\frac{\partial q_z^n(S^n, \boldsymbol{p}^n)}{\partial p_k^n(S^n, \boldsymbol{p}^n)} = \frac{q_k^n(S^n, \boldsymbol{p}^n)}{\mu_2} q_z^n(S^n, \boldsymbol{p}^n), z \in S^n, z \neq k, \tag{17}$$

$$\frac{\partial q_k^n(S^n, \boldsymbol{p}^n)}{\partial p_k^n} = -\frac{q_k^n(S^n, \boldsymbol{p}^n)}{\mu_2}[1 - q_k^n(S^n, \boldsymbol{p}^n)], k \in S^n, \tag{18}$$

$$\frac{\partial \tau^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)}{\partial p_k^R} = -\frac{q_k^R(S^R, \boldsymbol{p}^R)}{\mu_1}\tau^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)[1 - \tau^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)]. \tag{19}$$

The proofs of the propositions in Section 4 are for any given S^R; hence S^R is dropped from the argument list of the functions.

Proof of Proposition 1: The retailer's problem in (8) reduces to picking the price vector $\boldsymbol{p}^R$ to maximize its expected profit function $\Pi^R(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)$. We start by proving that any optimal price vector $\boldsymbol{p}^R$ must satisfy the first order conditions (FOC) of the retailer's objective function $\Pi^R(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)$.

Using the derivatives of the functions provided in (17), (18) and (19) it can be verified that the derivative of retailer's expected profit, $\Pi^R(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)$, given by (8), is:

$$\frac{\partial \Pi^R(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)}{\partial p_k^R} =$$

$$\frac{\tau^R(\boldsymbol{p}^M, \boldsymbol{p}^R) q_k^R(\boldsymbol{p}^R)}{\mu_2} \left[\begin{array}{l} \sum_{k \in S^R} (p_k^R - w_k - \gamma_k)\, q_k^R(\boldsymbol{p}^R) \left[1 - \frac{\mu_2(1 - \tau^R(\boldsymbol{p}^M, \boldsymbol{p}^R))}{\mu_1} \right] \\ -\, (p_k^R - w_k - \gamma_k) + \mu_2 \end{array} \right]. \tag{20}$$

It can be shown that (20) is negative (positive) when p_k^R is very large (very small), i.e.,

$$\lim_{p_k^R \to \infty} \frac{\partial \Pi^R(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)}{\partial p_k^R} < 0, \text{ and } \lim_{p_k^R \to -\infty} \frac{\partial \Pi^R(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)}{\partial p_k^R} > 0.$$

Hence, it is never optimal to set one of the prices at a boundary, and the optimal price vector must satisfy the FOC.

To conclude the proof of the proposition, we next prove that any optimal price vector $\boldsymbol{p}^R$ must satisfy the property that $p_k^R - w_k - \gamma_k$ is the same for all $k \in S^R$. Setting the derivative of $\Pi^R(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)$, provided in (20), equal to zero and re-arranging terms yields the following

FOC for $\Pi^R(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)$:

$$p_k^R - w_k - \gamma_k = \sum_{k \in S^R} (p_k^R - w_k - \gamma_k)\, q_k^R(\boldsymbol{p}^R) \left[1 - \frac{\mu_2(1 - \tau^R(\boldsymbol{p}^M, \boldsymbol{p}^R))}{\mu_1}\right] + \mu_2.$$

Observe that the right-hand side (RHS) of the above equation is the same for all $k \in S^R$. Therefore, $p_k^R - w_k - \gamma_k = m^R$ for all $k \in S^R$, where m^R is the effective profit margin for the retailer.

Proof of Proposition 2: In preparation for the proof, we note below that the derivative of (15) is:

$$\frac{\partial \tau^R(\boldsymbol{p}^M, m^R)}{\partial m^R} = -\frac{\tau^R(\boldsymbol{p}^M, m^R)}{\mu_1}[1 - \tau^R(\boldsymbol{p}^M, m^R)] < 0.$$

Using this derivative it can be verified that the derivative of $\Pi^R(\boldsymbol{p}^M, m^R)$, given in (16), with respect to m^R is given by:

$$\frac{\partial \Pi^R(\boldsymbol{p}^M, m^R)}{\partial m^R} = \tau^R(\boldsymbol{p}^M, m^R) \left[1 - \frac{m^R}{\mu_1}[1 - \tau^R(\boldsymbol{p}^M, m^R)]\right]. \tag{21}$$

Recall from Proposition 1 that the optimal prices must be interior solutions, that is, $-\infty < p_k^R < \infty$ for $k \in S^R$. In addition, observe from Proposition 1, the margin, m^R, and the price, p_k^R, differ by $w_k + \gamma_k$. Therefore, for finite $\boldsymbol{w}$, the optimal margin m^R must also be an interior solution, i.e., $-\infty < m^R < \infty$. Thus, m^R must satisfy the FOC of $\Pi^R(\boldsymbol{p}^M, m^R)$.

Setting the derivative of $\Pi^R(\boldsymbol{p}^M, m^R)$ (provided in (21)) equal to zero and rearranging terms, yields the following FOC for $\Pi^R(\boldsymbol{p}^M, m^R)$ with respect to m^R:

$$m^R = \frac{\mu_1}{1 - \tau^R(\boldsymbol{p}^M, m^R)}. \tag{22}$$

To conclude, we next prove that the m^R that satisfies this FOC is in fact unique. Substituting for $\tau^R(\boldsymbol{p}^M, m^R)$ in (22) with the expression provided in (15) and algebraically manipulating the terms, we can rewrite the above equality as:

$$\left(\frac{m^R}{\mu_1} - 1\right) \exp\left(\frac{m^R}{\mu_1} - 1\right) = \frac{\exp(-1)\left[\sum_{k \in S^R} \exp(\eta_k^R/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}.$$

Applying the definition of the Lambert-W function (provided in the preamble of Appendix A) and with some algebra we can rewrite the above expression as,

$$m^R(\boldsymbol{p}^M, \boldsymbol{w}) = \mu_1 \left[1 + W\left(\frac{\exp(-1)\left[\sum_{k \in S^R} \exp(\eta_k^R/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}\right)\right], \tag{23}$$

Observe that the right-hand side of the above equality does not depend on m^R and the left-hand side is strictly increasing in m^R. Therefore, there exists a unique m^R that satisfies the FOC.

Proof of Proposition 3: This proof will proceed in two steps. In Step 1, we will rewrite the manufacturer's profit, given in (10), as a function of the optimal effective margin chosen by the retailer in response to the manufacturer's wholesale and direct channel price vector, i.e., $m^R(\boldsymbol{p}^M, \boldsymbol{w})$ as provided in (23). In Step 2, we will use this version of the manufacturer's profit function to prove the properties that the manufacturer's optimal price vector must satisfy.

Step 1: We will show that the manufacturer's profit in (10) can be written as

$$\Pi^M(S^R, \boldsymbol{p}^M, \boldsymbol{w}, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})) = \tau^M(S^R, \boldsymbol{p}^M, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})) \sum_{k \in S^M} (p_k^M - c_k) q_k^M(\boldsymbol{p}^M)$$
$$+ \tau^R(S^R, \boldsymbol{p}^M, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})) \sum_{k \in S^R} (w_k - c_k)\theta_k q_k^R(S^R, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})),$$

where $\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$, $\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$ and $q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))$ are defined appropriately.

$$\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) = \frac{W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))}{1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))},$$

$$\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) = \frac{\left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}\right] \left[1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))\right]},$$

and

$$q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w})) = \frac{v_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\sum_{k \in S^n} v_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))} \text{ for all } k \in S^R.$$

Let $\Omega(\boldsymbol{p}^M, \boldsymbol{w})$ be defined as:

$$\Omega(\boldsymbol{p}^M, \boldsymbol{w}) := \frac{\exp(-1)\left[\sum_{k \in S^R} \exp([\alpha_k - w_k - \gamma_k]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}. \tag{24}$$

Using the expression above we can write $m^R(\boldsymbol{p}^M, \boldsymbol{w})$ in (23) as

$$m^R(\boldsymbol{p}^M, \boldsymbol{w}) = \mu_1 \left[1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))\right].$$

Observe from Proposition 2 that $\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) = 1 - \frac{\mu_1}{m^R(\boldsymbol{p}^M, \boldsymbol{w})}$. Substituting in this expression for $m^R(\boldsymbol{p}^M, \boldsymbol{w})$ with $\mu_1 \left[1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))\right]$ we can write $\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$ as,

$$\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) = \frac{W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))}{1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))}. \tag{25}$$

Next we recast $\tau^{M}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ in (4) in terms of $m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})$ instead of $\boldsymbol{p}^{R}$. To that end, we replace $v_k(p_k^{R})$ in the denominator of (4) with $v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))$, where $v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))$ is defined by (14):

$$\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})) = \frac{\left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^R} v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))\right]^{\mu_2/\mu_1}}. \tag{26}$$

Given that $m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}) = \mu_1 \left[1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))\right]$ the definition $\Omega(\boldsymbol{p}^{M}, \boldsymbol{w})$ and the fact that for any value $a > 0$ we have that $\exp(W(a)) = \frac{a}{W(a)}$, we can write the following equality:

$$\begin{aligned}
\left[\sum_{k\in S^R} v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))\right]^{\mu_2/\mu_1} &= \left[\sum_{k\in S^R} \exp([\eta_k^{R} - m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})]/\mu_2)\right]^{\mu_2/\mu_1} \\
&= \left[\sum_{k\in S^R} \exp([\eta_k^{R} - \mu_1 \left[1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))\right]/\mu_2)\right]^{\mu_2/\mu_1} \\
&= W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w})) \left[\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}\right].
\end{aligned}$$

Hence, we can write $\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))$, given by (26),

$$\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})) = \frac{\left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}\right]\left[1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))\right]}. \tag{27}$$

As for $q_k^{R}(\boldsymbol{p}^{R})$ given in (1), we use the definition of $v_k^{R}(m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}))$, in (14), to write,

$$q_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})) = \frac{v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\sum_{k\in S^n} v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))} \text{ for all } k \in S^R. \tag{28}$$

Provided the expressions above for the purchase probabilities, i.e. (25), (27) and (28), we can now recast the manufacturer's expected profit in (10) as,

$$\begin{aligned}
\Pi^{M}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}, m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})) &= \tau^{M}(S^R, \boldsymbol{p}^{M}, m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})) \sum_{k\in S^M} (p_k^{M} - c_k) q_k^{M}(\boldsymbol{p}^{M}) \\
&+ \tau^{R}(S^R, \boldsymbol{p}^{M}, m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})) \sum_{k\in S^R} (w_k - c_k)\theta_k q_k^{R}(S^R, m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})).
\end{aligned} \tag{29}$$

Step 2: To continue with the proof, let us now show that any optimal price vector $\boldsymbol{p}^{M}$ and $\boldsymbol{w}$ must satisfy the first order conditions (FOC) of the manufacturer's objective function $\Pi^{M}(\boldsymbol{p}^{M}, \boldsymbol{w}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))$, given by (29). Below are the partial derivatives of $\Omega(\boldsymbol{p}^{M}, \boldsymbol{w})$, given in

(24), $\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$, given in (25), $\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$, given in (27), and $q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))$, given in (28).

$$
\begin{aligned}
\frac{\partial \Omega(\boldsymbol{p}^M, \boldsymbol{w})}{\partial p_k^M} &= \frac{q_k^M(\boldsymbol{p}^M)\Omega(\boldsymbol{p}^M, \boldsymbol{w})}{\mu_1}\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))],\\
\frac{\partial \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial p_k^M} &= \frac{q_k^M(\boldsymbol{p}^M)}{\mu_1}\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))\\
&\quad \times [1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))]\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})),\\
\frac{\partial \tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial p_k^M} &= \frac{-q_k^M(\boldsymbol{p}^M)\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\mu_1}\\
&\quad \times \left[\begin{array}{l} 1 - \tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))] \\ + \tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) \end{array}\right],\\
\frac{\partial \Omega(\boldsymbol{p}^M, \boldsymbol{w})}{\partial w_k} &= \frac{-q_k^R(\boldsymbol{p}^R)\Omega(\boldsymbol{p}^M, \boldsymbol{w})}{\mu_1},\\
\frac{\partial \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial w_k} &= \frac{-q_k^R(\boldsymbol{p}^R)}{\mu_1}\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))]^2,\\
\frac{\partial \tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial w_k} &= \frac{q_k^R(\boldsymbol{p}^R)}{\mu_1}\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))\\
&\quad \times [1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))],\\
\frac{\partial q_l^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial w_k} &= \frac{q_l^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\mu_2}, l \in S^R, l \neq k,\\
\frac{\partial q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial w_k} &= -\frac{q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))]}{\mu_2}, k \in S^R.
\end{aligned}
$$

Using the derivatives above and some algebra one could verify that:

$$
\begin{aligned}
&\frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial p_k^M} = \frac{q_k^M(\boldsymbol{p}^M)\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\mu_1}\\
&\times \left[\begin{array}{l} \frac{\mu_2}{\mu_1}\left[\sum_{j \in S^M}(p_j^M - c_j)q_j^M(\boldsymbol{p}^M) - (p_k^M - c_k) + \mu_2\right] \\ - \sum_{j \in S^M}(p_j^M - c_j)q_j^M(\boldsymbol{p}^M)\left[1 - \tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))\left[\begin{array}{l} 1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w})) \\ - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) \end{array}\right]\right] \\ + \sum_{j \in S^R}(w_j - c_j)\theta_j q_j^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))] \end{array}\right], \qquad (30)
\end{aligned}
$$

and

$$\frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial w_k} = \frac{q_k^M(\boldsymbol{p}^M)\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\mu_1} \times \left[\begin{array}{l} \sum_{j \in S^M} (p_j^M - c_j) q_j^M(\boldsymbol{p}^M) \tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))] \\ - \sum_{j \in S^R} (w_j - c_j)\theta_j q_j^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))]^2 \\ - \frac{\mu_1}{\mu_2} \left[\sum_{j \in S^R} (w_j - c_j)\theta_j q_j^R(m^R(\boldsymbol{p}^M, \boldsymbol{w})) - (w_k - c_k - \mu_2)\theta_k \right] \end{array} \right] \tag{31}$$

It can be shown that (30) is negative (positive) when p_k^M is very large (very small), i.e.,

$$\lim_{p_k^M \to \infty} \frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial p_k^M} < 0 \text{ and } \lim_{p_k^M \to -\infty} \frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial p_k^M} > 0.$$

Hence, it is never optimal to set one of the wholesale prices at a boundary, and the optimal $\boldsymbol{p}^M$ must satisfy the FOC. Similarly, it can be shown that (31) is negative (positive) when w_k is very large (very small), i.e.,

$$\lim_{w_k \to \infty} \frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial w_k} < 0 \text{ and } \lim_{w_k \to -\infty} \frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial w_k} > 0.$$

Hence, it is never optimal to set one of the prices for the indirect channel at a boundary, and the optimal $\boldsymbol{w}$ must satisfy the FOC.

To conclude the proof of Proposition 3 (a), we next prove that any $\boldsymbol{p}^M$ that satisfy FOC of $\Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$ with respect to $\boldsymbol{p}^M$ must also satisfy the property that $p_k^M - c_k$ is the same for all $k \in S^M$. Setting (30) equal to zero and rearranging terms yields the following:

$$\begin{aligned} p_k^M - c_k = & \sum_{j \in S^M} (p_j^M - c_j) q_j^M(\boldsymbol{p}^M) + \mu_2 \\ & - \frac{\mu_1}{\mu_2} \sum_{j \in S^M} (p_j^M - c_j) q_j^M(\boldsymbol{p}^M) \left[\begin{array}{l} 1 - \tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))] \\ - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) \end{array} \right] \\ & + \frac{\mu_1}{\mu_2} \sum_{j \in S^R} (w_j - c_j)\theta_j q_j^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))]. \end{aligned}$$

Observe that the right-hand side (RHS) of the above equation is the same for all $k \in S^M$. Therefore, $p_k^M - c_k$ is the same for all $k \in S^M$. This concludes the proof of Proposition 3 (a).

Likewise, setting (31) equal to zero and rearranging terms yields the following:

$$
\begin{aligned}
(w_k - c_k - \mu_2)\theta_k = &\frac{\mu_2}{\mu_1} \sum_{j \in S^M} (p_j^{M} - c_j) q_j^{M}(\boldsymbol{p}^{M}) \tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))[1 - \tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))] \\
&- \frac{\mu_2}{\mu_1} \sum_{j \in S^R} (w_j - c_j)\theta_j q_j^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))[1 - \tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))]^2 \\
&- \sum_{j \in S^R} (w_j - c_j)\theta_j q_j^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})).
\end{aligned}
$$

Observe that the RHS of the above equation is the same for all $k \in S^R$. This implies that $(w_k - c_k - \mu_2)\theta_k$ is the same for all $k \in S^R$. Therefore, the optimization over prices reduces to optimizing over two margins, m^{M} and $\overline{w}$, where $m^{M} := p_k^{M} - c_k$ for all $k \in S^M$, and $\overline{w} := (w_k - c_k - \mu_2)\theta_k$ for all $k \in S^R$.

Proof of Proposition 4: Recall the definition of θ_k provided in (11), i.e. $\theta_k := \sigma_k (z_k + I_N(z_k)) + 1$. We now replace $I_N(z_k)$ in θ_k by its definition, $I_N(z_k) := \phi_N(z_k) - z_k(1 - \Phi_N(z_k))$, where $\phi_N(\cdot)$ and $\Phi_N(\cdot)$ are the standard normal density and distribution functions, respectively. After replacing $I_N(z_k)$ and with some algebraic manipulation we obtain

$$\theta_k := \sigma_k (\phi_N(z_k) + z_k \Phi_N(z_k)) + 1.$$

We note that $\phi_N(z_k) + z_k\Phi_N(z_k) > 0$, because $\lim_{z_k \to -\infty} [\phi_N(z_k) + z_k\Phi_N(z_k)] = 0$ and $\phi_N(z_k) + z_k\Phi_N(z_k)$ increases in z_k. Therefore, observe from the above equality that θ_k is increasing in σ_k. This fact will be useful in the rest of the proof.

Now recall from Proposition 3 that $\overline{w} = (w_k - c_k - \mu_2)\theta_k$ for all $k \in S^R$. Given that the proposition assumes that all variants are the same in all respects but σ_k, notice from the expression that variants with higher θ_k will have lower w_k. Recall that θ_k is increasing in σ_k. Hence, higher σ_k implies lower w_k.

Proofs for Section 5

For the purposes of this section, the retailer's assortment, S^R, is fixed and hence we drop S^R from the argument list of the functions. We first provide and prove three lemmas that will be used in the proofs of the propositions in Section 5.

Lemma 1. *The manufacturer's profit, $\Pi^M(\boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R})$ in (10), can be written as:*

$$
\begin{aligned}
&\Pi^M(m^{M}, \overline{w}, m^{R}(m^{M}, \overline{w})) = \\
&\tau^{M}(m^{M}, m^{R}(m^{M}, \overline{w}))m^{M} + \tau^{R}(m^{M}, m^{R}(m^{M}, \overline{w})) \sum_{k \in S^R} (\overline{w} + \theta_k \mu_2) q_k^{R}(m^{R}(m^{M}, \overline{w})),
\end{aligned}
$$

where

$$v_k^{M}(m^{M}) = \exp([\alpha_k - c_k - m^{M}]/\mu_2) \text{ for } k \in S^M, \tag{32}$$

$$v_k^{R}(m^{R}(m^{M},\overline{w})) = \exp([\alpha_k - \overline{w}/\theta_k - c_k - \mu_2 - \gamma_k - m^{R}(m^{M},\overline{w})]/\mu_2) \text{ for } k \in S^R, \tag{33}$$

$$\tau^{R}(m^{M}, m^{R}(m^{M},\overline{w})) = \frac{\left[\sum_{k\in S^R} v_k^{R}(m^{R}(m^{M},\overline{w}))\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^R} v_k^{R}(m^{R}(m^{M},\overline{w}))\right]^{\mu_2/\mu_1}}, \tag{34}$$

$$\tau^{M}(m^{M}, m^{R}(m^{M},\overline{w})) = \frac{\left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^R} v_k^{R}(m^{R}(m^{M},\overline{w}))\right]^{\mu_2/\mu_1}} \tag{35}$$

$$\text{and } q_k^{R}(m^{R}(m^{M},\overline{w})) = \frac{v_k^{R}(m^{R}(m^{M},\overline{w}))}{\sum_{j\in S^R} v_j^{R}(m^{R}(m^{M},\overline{w}))}. \tag{36}$$

Proof of Lemma 1: Let $v_k^{M}(m^{M})$ and $v_k^{R}(m^{R}(m^{M},\overline{w}))$ be as defined in the statement of the lemma. Using Proposition 3 (a) it can be verified that

$$v_k^{M}(m^{M}) = v_k(p_k^{M}), \tag{37}$$

where $v_k(p_k^n)$ is as defined in (2). Similarly, using Proposition 3 (b) it can be verified that

$$v_k^{R}(m^{R}(m^{M},\overline{w})) = v_k(p_k^{R}). \tag{38}$$

Using the equalities in (37) and (38) we will make the following three observations:

Observation 1: Recall the expression for $\tau^{R}(\boldsymbol{p}^{M},\boldsymbol{p}^{R})$ given in (4), i.e.

$$\tau^{R}(\boldsymbol{p}^{M},\boldsymbol{p}^{R}) = \frac{\left[\sum_{k\in S^R} v_k(p_k^{R})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^R} v_k(p_k^{R})\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}.$$

Using the equalities in (37) and (38), observe that

$$\tau^{R}(\boldsymbol{p}^{M},\boldsymbol{p}^{R}) = \tau^{R}(m^{M}, m^{R}(m^{M},\overline{w})),$$

where $\tau^{R}(m^{M}, m^{R}(m^{M},\overline{w}))$ is as defined in the statement of the lemma.

Observation 2: Recall the expression for $\tau^{M}(\boldsymbol{p}^{M},\boldsymbol{p}^{R})$ given in (4), i.e.

$$\tau^{M}(\boldsymbol{p}^{M},\boldsymbol{p}^{R}) = \frac{\left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^R} v_k(p_k^{R})\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}.$$

Observe that $\tau^{M}(\boldsymbol{p}^{M},\boldsymbol{p}^{R}) = \tau^{M}(m^{M}, m^{R}(m^{M},\overline{w}))$, where $\tau^{M}(m^{M}, m^{R}(m^{M},\overline{w}))$ is as defined in the statement of the lemma.

Observation 3: Recall the expression for $q_k^R(\boldsymbol{p}^R)$ in (1), i.e.

$$q_k^R(\boldsymbol{p}^R) = \frac{v_k(p_k^R)}{\sum_{k\in S^R} v_k(p_k^R)} \text{ for all } k \in S^R.$$

Observe that $q_k^R(\boldsymbol{p}^R) = q_k^R(m^R(m^M, \overline{w}))$, where $q_k^R(m^R(m^M, \overline{w}))$ is as defined in the statement of the lemma.

Using Proposition 3, we make the following two observations.

Observation 4: From Proposition 3 (a), we have that $p_k^M = m^M + c_k$. Hence, observe that we can rewrite the term $\sum_{k\in S^M}(p_k^M - c_k)q_k^M(\boldsymbol{p}^M)$ in (10) as just m^M (this is because $\sum_{k\in S^M} q_k^M(\boldsymbol{p}^M) = 1$).

Observation 5: From Proposition 3 (b), we have that $w_k = \overline{w}/\theta_k + c_k + \mu_2$. Hence, observe that we can write the term $\sum_{k\in S^R}(w_k - c_k)\theta_k q_k^R(\boldsymbol{p}^R)$ in (10) as $\sum_{k\in S^R}(\overline{w} + \theta_k\mu_2)q_k^R(\boldsymbol{p}^R)$.

Recall the definition of $\Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R)$ in (10), i.e.

$$\Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, \boldsymbol{p}^R) =$$
$$\tau^M(\boldsymbol{p}^M, \boldsymbol{p}^R) \sum_{k\in S^M} (p_k^M - c_k)q_k^M(\boldsymbol{p}^M) + \tau^R(\boldsymbol{p}^M, \boldsymbol{p}^R) \sum_{k\in S^R} (w_k - c_k)\theta_k q_k^R(\boldsymbol{p}^R).$$

We now use observations 1 through 5 and substitute in the above equation: $\tau^M(m^M, m^R(m^M, \overline{w}))$ for $\tau^M(\boldsymbol{p}^M, \boldsymbol{p}^R)$, $\tau^R(m^M, m^R(m^M, \overline{w}))$ for $\tau^R(\boldsymbol{p}^M, \boldsymbol{p}^R)$, $q_k^R(m^R(m^M, \overline{w}))$ for $q_k^R(\boldsymbol{p}^R)$, $\sum_{k\in S^R}(\overline{w} + \theta_k\mu_2)q_k^R(m^R(m^M, \overline{w}))$ for $\sum_{k\in S^R}(w_k - c_k)\theta_k q_k^R(\boldsymbol{p}^R)$ and m^M for $\sum_{k\in S^M}(p_k^M - c_k)q_k^M(\boldsymbol{p}^M)$. This yields

$$\Pi^M(m^M, \overline{w}, m^R(m^M, \overline{w})) = \tau^M(m^M, m^R(m^M, \overline{w}))m^M + \tau^R(m^M, m^R(m^M, \overline{w})) \sum_{k\in S^R} (\overline{w} + \theta_k\mu_2)q_k^R(m^R(m^M, \overline{w})). \quad (39)$$

Lemma 2. *The manufacturer's profit function,* $\Pi^M(m^M, \overline{w}, m^R(m^M, \overline{w}))$*, given in Lemma 1, can be written as:*

$$\Pi^M(m^M, \overline{w}) = \widehat{\tau}^M(m^M, \overline{w})m^M + \widehat{\tau}^R(m^M, \overline{w}) \sum_{k\in S^R} (\overline{w} + \theta_k\mu_2)\widehat{q}_k^R(m^M, \overline{w}),$$

where

$$\Omega(m^M,\overline{w}) = \frac{\exp(-1)\left[\sum_{j\in S^R}\exp([\alpha_k - c_k - \gamma_k - \mu_2 - \overline{w}/\theta_j]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^M(m^M)\right]^{\mu_2/\mu_1}},$$

$$v_k^M(m^M) = \exp([\alpha_k - c_k - m^M]/\mu_2) \text{ for } k \in S^M,$$

$$\widehat{v}_k^R(m^M,\overline{w}) = \exp([\alpha_k - \overline{w}/\theta_k - c_k - \mu_2 - \gamma_k - \mu_1[1 + W(\Omega(m^M,\overline{w}))]]/\mu_2),$$

$$\widehat{\tau}^R(m^M,\overline{w}) = \frac{W(\Omega(m^M,\overline{w}))}{1 + W(\Omega(m^M,\overline{w}))},$$

$$\widehat{\tau}^M(m^M,\overline{w}) = \frac{\left[\sum_{k\in S^M} v_k^M(m^M)\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^M(m^M)\right]^{\mu_2/\mu_1}\right]\left[1 + W(\Omega(m^M,\overline{w}))\right]},$$

$$\text{and } \widehat{q}_k^R(m^M,\overline{w}) = \frac{\widehat{v}_k^R(m^M,\overline{w})}{\sum_{j\in S^R}\widehat{v}_j^R(m^M,\overline{w})}.$$

Proof of Lemma 2: Let us first define

$$\Omega(m^M,\overline{w}) := \frac{\exp(-1)\left[\sum_{j\in S^R}\exp([\alpha_k - c_k - \gamma_k - \mu_2 - \overline{w}/\theta_j]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^M(m^M)\right]^{\mu_2/\mu_1}}. \tag{40}$$

Recall from Proposition 2 that

$$m^R(m^M,\overline{w}) = \frac{\mu_1}{1 - \tau^R(m^M, m^R(m^M,\overline{w}))}.$$

Replacing $\tau^R(m^M, m^R(m^M,\overline{w}))$ in this expression with its definition in Lemma 1 we obtain:

$$\begin{aligned}\frac{m^R(m^M,\overline{w})}{\mu_1} &= \frac{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^M(m^M)\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^R} v_k^R(m^R(m^M,\overline{w}))\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^M(m^M)\right]^{\mu_2/\mu_1}} \\ &= 1 + \frac{\left[\sum_{j\in S^R}\exp([\alpha_k - c_k - \gamma_k - \mu_2 - \overline{w}/\theta_j]/\mu_2)\right]^{\mu_2/\mu_1}\exp(-m^R/\mu_1)}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^M(m^M)\right]^{\mu_2/\mu_1}}.\end{aligned}$$

With some algebraic manipulation we now obtain:

$$\begin{aligned}&\left[\frac{m^R(m^M,\overline{w})}{\mu_1} - 1\right]\exp\left(\frac{m^R(m^M,\overline{w})}{\mu_1} - 1\right) = \\ &\qquad \frac{\exp(-1)\left[\sum_{j\in S^R}\exp([\alpha_k - c_k - \gamma_k - \mu_2 - \overline{w}/\theta_j]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^M(m^M)\right]^{\mu_2/\mu_1}}.\end{aligned}$$

Observe that the right-hand side of the expression above is the same as $\Omega(m^M,\overline{w})$. Using the Lambert-W function provided in the preamble of Appendix A, it could be shown that the expression above can be written as:

$$\frac{m^R(m^M,\overline{w})}{\mu_1} - 1 = W(\Omega(m^M,\overline{w})). \tag{41}$$

We now use the expression above written in the form of

$$m^{R}(m^{M},\overline{w})=\mu_1[1+W(\Omega(m^{M},\overline{w}))] \tag{42}$$

to make some observations.

Observation 1: Recall from Proposition 2 that $\tau^{R}(m^{M},m^{R}(m^{M},\overline{w}))=1-\frac{\mu_1}{m^{R}(m^{M},\overline{w})}$. Replacing $m^{R}(m^{M},\overline{w})$ in this expression by the definition of $m^{R}(m^{M},\overline{w})$ in (42), we can observe that:

$$\widehat{\tau}^{R}(m^{M},\overline{w})=\tau^{R}(m^{M},m^{R}(m^{M},\overline{w})),$$

where $\widehat{\tau}^{R}(m^{M},\overline{w})$ is defined in the statement of the lemma.

Observation 2: Recall the expression for $v_k^{R}(m^{R}(m^{M},\overline{w}))$ defined in Lemma 1, i.e.

$$v_k^{R}(m^{R}(m^{M},\overline{w}))=\exp([\alpha_k-\overline{w}/\theta_k-c_k-\mu_2-\gamma_k-m^{R}(m^{M},\overline{w})]/\mu_2).$$

Substituting above for $m^{R}(m^{M},\overline{w})$ with $\mu_1[1+W(\Omega(m^{M},\overline{w}))]$ we obtain:

$$v_k^{R}(m^{R}(m^{M},\overline{w}))=\exp([\alpha_k-\overline{w}/\theta_k-c_k-\mu_2-\gamma_k-\mu_1[1+W(\Omega(m^{M},\overline{w}))]]/\mu_2).$$

Observe from above that $v_k^{R}(m^{R}(m^{M},\overline{w}))=\widehat{v}_k^{R}(m^{M},\overline{w})$ where $\widehat{v}_k^{R}(m^{M},\overline{w})$ is as defined in the lemma.

Observation 3: Recall from Proposition 3 that $p_k^{M}=m^{M}+c_k$. Replacing p_k^{M} with $m^{M}+c_k$ in the expression for $v_k(p_k^{M})$ provided in (2), we conclude that $v_k(p_k^{M})=v_k^{M}(m^{M})$. Using Proposition 3 we recall also that $w_k=\overline{w}_k/\theta_k+c_k+\mu_2$. Replacing w_k with $\overline{w}_k/\theta_k+c_k+\mu_2$ and $v_k(p_k^{M})$ with $v_k^{M}(m^{M})$ in the expression for $\Omega(\boldsymbol{p}^{M},\boldsymbol{w})$ provided in (24) we conclude that $\Omega(\boldsymbol{p}^{M},\boldsymbol{w})=\Omega(m^{M},\overline{w})$, where $\Omega(m^{M},\overline{w})$ is provided in (40).

Recall the expression for $\tau^{M}(\boldsymbol{p}^{M},m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))$ provided in (27), i.e.

$$\tau^{M}(\boldsymbol{p}^{M},m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))=\frac{\left[\sum_{k\in S^M}v_k(p_k^{M})\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0)+\left[\sum_{k\in S^M}v_k(p_k^{M})\right]^{\mu_2/\mu_1}\right][1+W(\Omega(\boldsymbol{p}^{M},\boldsymbol{w}))]}.$$

Notice that replacing $v_k(p_k^{M})$ with $v_k^{M}(m^{M})$ and $\Omega(\boldsymbol{p}^{M},\boldsymbol{w})$ with $\Omega(m^{M},\overline{w})$ in the expression above we can conclude that

$$\tau^{M}(m^{M},m^{R}(m^{M},\overline{w}))=\widehat{\tau}^{M}(m^{M},\overline{w}).$$

Observation 4: Recall also the expression for $q_k^{R}(m^{R})$ in Lemma 1, i.e.

$$q_k^{R}(m^{R}(m^{M},\overline{w}))=\frac{v_k^{R}(m^{R}(m^{M},\overline{w}))}{\sum_{j\in S^R}v_j^{R}(m^{R}(m^{M},\overline{w}))}.$$

Using Observation 2, we can substitute $\widehat{v}_k^R(m^M, \overline{w})$ for $v_k^R(m^R(m^M, \overline{w}))$, which will allow us to observe that

$$q_k^R(m^R(m^M, \overline{w})) = \widehat{q}_k^R(m^M, \overline{w}).$$

Using observations 1 through 5, we now replace $\tau^M(m^M, m^R)$ with $\widehat{\tau}^M(m^M, \overline{w})$, $\tau^R(m^M, m^R)$ with $\widehat{\tau}^R(m^M, \overline{w})$ and $q_k^R(m^R)$ with $\widehat{q}_k^R(\Omega(m^M, \overline{w}))$ in the manufacturer's profit function $\Pi^M(m^M, \overline{w}, m^R)$ in (39). This step yields

$$\Pi^M(m^M, \overline{w}) = \widehat{\tau}^M(m^M, \overline{w})m^M + \widehat{\tau}^R(m^M, \overline{w}) \sum_{k \in S^R} (\overline{w} + \theta_k \mu_2)\widehat{q}_k^R(m^M, \overline{w}). \tag{43}$$

Lemma 3. *Consider a manufacturer selling through the direct channel only. Let $m^M_{k,direct} := p_k^M - c_k$ denote the gross margin for variant $k \in S^M$. Then, any price vector that is optimal for the manufacturer is such that all variants have the same gross margin, i.e. $m^M_{k,direct} = m^M_{direct}$ for all $k \in S^M$. Furthermore, the manufacturer's optimal direct channel margin is the unique value of m^M_{direct} that satisfies*

$$m^M_{direct} = \frac{\mu_1}{1 - \tau^M_{direct}(m^M_{direct})}, \tag{44}$$

where

$$\tau^M_{direct}(m^M_{direct}) = \frac{\left[\sum_{k \in S^M} v_k^M(m^M_{direct})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k^M(m^M_{direct})\right]^{\mu_2/\mu_1}}, \tag{45}$$

and $v_k^M(m^M_{direct}) = \exp([\alpha_k - c_k - m^M_{direct}]/\mu_2)$.

Proof of Lemma 3: For a manufacturer selling only through a direct channel, the probability that a customer purchases from the direct channel, denoted with $\tau^M_{direct}(\boldsymbol{p}^M)$, will be given by:

$$\tau^M_{direct}(\boldsymbol{p}^M) = \frac{\left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}. \tag{46}$$

Using this probability, the manufacturer's expected profit will then be given by:

$$\Pi^M_{direct}(\boldsymbol{p}^M) = \tau^M_{direct}(\boldsymbol{p}^M) \sum_{k \in S^M} (p_k^M - c_k) q_k^M(\boldsymbol{p}^M). \tag{47}$$

Taking the derivative of (47) yields,

$$\frac{\partial \Pi^M_{direct}(\boldsymbol{p}^M)}{\partial p_k^M} = \frac{\tau^M_{direct}(\boldsymbol{p}^M) q_k^M(\boldsymbol{p}^M)}{\mu_2} \left[\begin{array}{l} - \displaystyle\sum_{j \in S^M} (p_j^M - c_j) q_j^M(\boldsymbol{p}^M) \left(\frac{\mu_2[1 - \tau^M_{direct}(\boldsymbol{p}^M)]}{\mu_1} - 1 \right) \\ - p_k^M + c_k - 1 \end{array} \right]. \tag{48}$$

It can be shown that:
$\lim_{p_k^M \to -\infty} \frac{\partial \Pi^M_{direct}(\boldsymbol{p}^M)}{\partial p_k^M} > 0$ and $\lim_{p_k^M \to \infty} \frac{\partial \Pi^M_{direct}(\boldsymbol{p}^M)}{\partial p_k^M} < 0$.
Hence, $\boldsymbol{p}^M$ must satisfy FOC, where the FOC are obtained by setting (48) equal to zero yielding:

$$p_k^M - c_k = \sum_{j \in S^M} (p_j^M - c_j) q_j^M(\boldsymbol{p}^M) \left(1 - \frac{\mu_2 [1 - \tau^M_{direct}(\boldsymbol{p}^M)]}{\mu_1}\right) - 1. \tag{49}$$

Observe from above that the right-hand side is the same for all $k \in S^M$. Let m^M_{direct} be the profit margin for the manufacturer, i.e. $m^M_{direct} := p_k^M - c_k$. Using this margin definition we now write the manufacturer's purchasing probability and expected profit as,

$$\tau^M_{direct}(m^M_{direct}) = \frac{\left[\sum_{k \in S^M} v_k^M(m^M_{direct})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k^M(m^M_{direct})\right]^{\mu_2/\mu_1}}, \tag{50}$$

where $v_k(m^M_{direct}) := \exp([\alpha_k - c_k - m^M_{direct}]/\mu_2)$, and

$$\Pi^M_{direct}(m^M_{direct}) = \tau^M_{direct}(m^M_{direct}) m^M_{direct}. \tag{51}$$

Recall that m^M_{direct} must satisfy FOC. Taking the derivative of (51) and setting it equal to zero yields the following FOC for the profit margin:

$$m^M_{direct} = \frac{\mu_1}{1 - \tau^M_{direct}(m^M_{direct})}. \tag{52}$$

Proof of Proposition 5: Let m^M_{direct} be the manufacturer's optimal direct channel margin when selling through the direct channel only, and let m^M_{dual} be the manufacturer's optimal direct channel margin when selling through dual channels, as required by the proposition. We first derive conditions that m^M_{dual} must satisfy for the dual-channel setting and provide an expression for the m^M_{dual} that satisfies those conditions.

In particular, we show that the optimal m^M for the dual-channel is such that:

$$\begin{aligned} m^M_{dual} = &\mu_1 \left[1 + \frac{\left[\sum_{k \in S^M} v_k^M(m^M_{dual})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)}\right] \\ &\times \left[1 + \sum_{j \in S^R} \theta_j \widehat{q}_j^R(m^M_{dual}, \overline{w}) W(\Omega(m^M_{dual}, \overline{w}))\right]. \end{aligned} \tag{53}$$

Recall from the proof of Proposition 3 that any optimal price vector for the manufacturer must satisfy FOCs. Therefore, m^M and $\overline{w}$ will also satisfy FOCs. We next use the FOCs to derive a simpler condition that the optimal m^M must satisfy. This proof uses the functions defined in the statement of Lemma 2.

Provided next are the derivatives of the functions $\Omega(m^M, \overline{w})$, $\widehat{\tau}^R(m^M, \overline{w})$, $\widehat{\tau}^M(m^M, \overline{w})$ and $\widehat{q}_k^R(m^M, \overline{w})$, all defined in Lemma 2, with respect to m^M and $\overline{w}$:

$$
\begin{aligned}
\frac{\partial \Omega(m^M, \overline{w})}{\partial m^M} &= \frac{\Omega(m^M, \overline{w})}{\mu_1} \widehat{\tau}^M(m^M, \overline{w})[1 + W(\Omega(m^M, \overline{w}))], \\
\frac{\partial \widehat{\tau}^R(m^M, \overline{w})}{\partial m^M} &= \frac{\widehat{\tau}^M(m^M, \overline{w}) \widehat{\tau}^R(m^M, \overline{w})[1 - \widehat{\tau}^R(m^M, \overline{w})]}{\mu_1}, \\
\frac{\partial \widehat{\tau}^M(m^M, \overline{w})}{\partial m^M} &= \frac{-\widehat{\tau}^M(m^M, \overline{w})}{\mu_1} \begin{bmatrix} 1 - \widehat{\tau}^M(m^M, \overline{w})[1 + W(\Omega(m^M, \overline{w}))] \\ + \widehat{\tau}^M(m^M, \overline{w}) \widehat{\tau}^R(m^M, \overline{w}) \end{bmatrix}, \\
\frac{\partial \Omega(m^M, \overline{w})}{\partial \overline{w}} &= -\frac{\Omega(m^M, \overline{w})}{\mu_1} \sum_{j \in S^R} \frac{\widehat{q}_j^R(m^M, \overline{w})}{\theta_j}, \\
\frac{\partial \widehat{\tau}^R(m^M, \overline{w})}{\partial \overline{w}} &= \frac{-\widehat{\tau}^R(m^M, \overline{w})}{\mu_1} [1 - \widehat{\tau}^R(m^M, \overline{w})]^2 \sum_{j \in S^R} \frac{\widehat{q}_j^R(m^M, \overline{w})}{\theta_j}, \\
\frac{\partial \widehat{\tau}^M(m^M, \overline{w})}{\partial \overline{w}} &= \frac{\widehat{\tau}^M(m^M, \overline{w}) \widehat{\tau}^R(m^M, \overline{w})[1 - \widehat{\tau}^R(m^M, \overline{w})]}{\mu_1} \sum_{j \in S^R} \frac{\widehat{q}_j^R(m^M, \overline{w})}{\theta_j},
\end{aligned}
$$

$$
\frac{\partial \widehat{q}_k^R(m^M, \overline{w})}{\partial \overline{w}} = -\frac{\widehat{q}_k^R(m^M, \overline{w})}{\mu_2} \left[\frac{1}{\theta_k} - \sum_{j \in S^R} \frac{\widehat{q}_j^R(m^M, \overline{w})}{\theta_j} \right].
$$

Using the derivatives above, it can be shown that the derivatives of $\Pi^M(m^M, \overline{w})$, given in Lemma 2 are:

$$
\begin{aligned}
\frac{\partial \Pi^M(m^M, \overline{w})}{\partial m^M} &= \widehat{\tau}^M(m^M, \overline{w}) \\
&\times \begin{bmatrix} 1 - \dfrac{m^M}{\mu_1} [1 + \widehat{\tau}^M(m^M, \overline{w}) (-1 - W(\Omega(m^M, \overline{w})) + \widehat{\tau}^R(m^M, \overline{w}))] \\ + \displaystyle\sum_{j \in S^R} (\overline{w} + \theta_j \mu_2) \widehat{q}_j^R(m^M, \overline{w}) \frac{W(\Omega(m^M, \overline{w}))}{\mu_1 [1 + W(\Omega(m^M, \overline{w}))]^2} \end{bmatrix},
\end{aligned} \tag{54}
$$

and

$$
\begin{aligned}
\frac{\partial \Pi^M(m^M, \overline{w})}{\partial \overline{w}} &= \sum_{j \in S^R} \frac{\widehat{q}_j^R(m^M, \overline{w})}{\theta_j} \widehat{\tau}^R(m^M, \overline{w}) \\
&\times \begin{bmatrix} \left[m^M \widehat{\tau}^M(m^M, \overline{w}) - \dfrac{\overline{w}}{1 + W(\Omega(m^M, \overline{w}))} \right] \dfrac{1}{\mu_1 [1 + W(\Omega(m^M, \overline{w}))]} \\ + \displaystyle\sum_{j \in S^R} \theta_j \widehat{q}_j^R(m^M, \overline{w}) \left[1 - \dfrac{\mu_2}{\mu_1 [1 + W(\Omega(m^M, \overline{w}))]^2} \right] \end{bmatrix}.
\end{aligned} \tag{55}
$$

To obtain the FOC for the manufacturer, we set (54) and (55) equal to zero and rearrange terms which yields,

$$\frac{1}{W(\Omega(m^M,\overline{w}))}\left[1-\frac{m^M}{\mu_1}\left[1-\widehat{\tau}^M(m^M,\overline{w})\left(1+W(\Omega(m^M,\overline{w}))\right)\right]\right]$$
$$=\frac{1}{\mu_1[1+W(\Omega(m^M,\overline{w}))]}\left[m^M\widehat{\tau}^M(m^M,\overline{w})-\frac{\sum_{j\in S^R}(\overline{w}+\theta_j\mu_2)\widehat{q}_j^R(m^M,\overline{w})}{[1+W(\Omega(m^M,\overline{w}))]}\right], \tag{56}$$

and

$$\frac{1}{\mu_1[1+W(\Omega(m^M,\overline{w}))]}\left[m^M\widehat{\tau}^M(m^M,\overline{w})-\frac{\sum_{j\in S^R}(\overline{w}+\theta_j\mu_2)\widehat{q}_j^R(m^M,\overline{w})}{[1+W(\Omega(m^M,\overline{w}))]}\right]$$
$$=-\sum_{j\in S^R}\theta_j\widehat{q}_j^R(m^M,\overline{w}). \tag{57}$$

Notice from the above two expressions that the RHS of (56) and the LHS of (57) are the same. Hence, to simplify the comparison between models, we replace the LHS of (56) with the RHS of (57) to obtain the following condition for m^M:

$$m^M=\frac{\mu_1\left[1+\sum_{j\in S^R}\theta_j\widehat{q}_j^R(m^M,\overline{w})W(\Omega(m^M,\overline{w}))\right]}{\left[1-\widehat{\tau}^M(m^M,\overline{w})\left[1+W(\Omega(m^M,\overline{w}))\right]\right]}. \tag{58}$$

For exposition purposes, we introduce the subscript *dual* to the functions and variables for the problem where the manufacturer sells through two channels. Let m^M_{dual} be the direct channel margin that satisfies the FOCs for the dual channel problem. Replacing $\widehat{\tau}^M(m^M,\overline{w})$ in (58) with its definition in Lemma 2, we can write

$$m^M_{dual}=\mu_1\left[1+\frac{\left[\sum_{k\in S^M}v_k^M(m^M_{dual})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)}\right]$$
$$\times\left[1+\sum_{j\in S^R}\theta_j\widehat{q}_j^R(m^M_{dual},\overline{w})W(\Omega(m^M_{dual},\overline{w}))\right].$$

Through algebra, we can rewrite the above condition as follows:

$$\frac{m^M_{dual}}{\mu_1\left[1+\frac{\left[\sum_{k\in S^M}v_k^M(m^M_{dual})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)}\right]}-1=\sum_{j\in S^R}\theta_j\widehat{q}_j^R(m^M_{dual},\overline{w})W(\Omega(m^M_{dual},\overline{w})). \tag{59}$$

Next, using the condition above we compare m^M_{dual} with m^M_{direct}.

Recall from the proof of Lemma 3 that m^M_{direct} must satisfy the expression in (52). Substituting in (52) for $\tau^M_{direct}(m^M_{direct})$ with its definition given in (50) and with some algebraic manipulation, yields the following condition:

$$\frac{m^M_{direct}}{\mu_1\left[1+\frac{\left[\sum_{k\in S^M}v_k^M(m^M_{direct})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)}\right]}-1=0. \tag{60}$$

Recall that $v_k^M(m^M)$ is decreasing in m^M. Given that (i) the left-hand side of the expressions in (59) and (60) are increasing in m^M, and (ii) the expression $\sum_{j\in S^R}\theta_j\widehat{q}_j^R(m_{dual}^M,\overline{w})W(\Omega(m_{dual}^M,\overline{w}))$ is always positive, we can now conclude that the m_{direct}^M that satisfies (60) is smaller than the m_{dual}^M that satisfies (59).

Proof of Proposition 6: Let $\Pi_{direct}^M(m_{direct}^M)$ be the manufacturer's optimal expected profit when selling through the direct channel only and $\Pi_{dual}^M(m_{dual}^M,\overline{w})$ be the manufacturer's optimal expected profit when selling through dual channels.

Recall the expression for m_{direct}^M provided in Lemma 3. Rearranging its terms, we obtain that $\tau_{direct}^M(m_{direct}^M) = 1 - \frac{\mu_1}{m_{direct}^M}$. We now replace $\tau_{direct}^M(m_{direct}^M)$ with $1 - \frac{\mu_1}{m_{direct}^M}$ in $\Pi_{direct}^M(m_{direct}^M)$, (51), to obtain the manufacturer's optimal expected profit, i.e.

$$\Pi_{direct}^M(m_{direct}^M) = m_{direct}^M - \mu_1. \tag{61}$$

This last expression for $\Pi_{direct}^M(m_{direct}^M)$ is going to be used later in this proof to compare it with the manufacturer's optimal profit for the dual channel setting.

Recall the expression for $\Pi_{dual}^M(m_{dual}^M,\overline{w})$ given in Lemma 2. To obtain an expression for the optimal expected profit for the manufacturer when selling through dual channels, we first rewrite the FOC of the manufacturer with respect to m_{dual}^M, given in (56). Observe we can rewrite (56) with some algebra to obtain:

$$\begin{aligned}&m_{dual}^M - \mu_1 + W(\Omega(m_{dual}^M,\overline{w}))\left[m_{dual}^M[1-\widehat{\tau}^M(m_{dual}^M,\overline{w})(1+W(\Omega(m_{dual}^M,\overline{w})))] - \mu_1\right] = \\ &\widehat{\tau}^M(m_{dual}^M,\overline{w})m_{dual}^M + \widehat{\tau}^R(m_{dual}^M,\overline{w})\sum_{k\in S^R}(\overline{w}+\theta_k\mu_2)\widehat{q}_k^R(m_{dual}^M,\overline{w}).\end{aligned}$$

Notice that the right-hand side (RHS) of the equation above corresponds to the definition of $\Pi_{dual}^M(m_{dual}^M,\overline{w})$ in (43). We can now replace the terms that correspond to the RHS of the equation above with the function $\Pi_{dual}^M(m_{dual}^M,\overline{w})$, to obtain,

$$\begin{aligned}\Pi_{dual}^M(m_{dual}^M,\overline{w}) &= m_{dual}^M - \mu_1 + W(\Omega(m_{dual}^M,\overline{w})) \\ &\times\left[m_{dual}^M[1-\widehat{\tau}^M(m_{dual}^M,\overline{w})(1+W(\Omega(m_{dual}^M,\overline{w})))] - \mu_1\right].\end{aligned} \tag{62}$$

Given the fact that $m_{direct}^M \leq m_{dual}^M$, as stated on Proposition 5, what is left to show in order to prove that the expression in (62) is greater than or equal to the expression in (61), is that

$$m_{dual}^M(1-\widehat{\tau}^M(m_{dual}^M,\overline{w})[1+W(\Omega(m_{dual}^M,\overline{w}))]) - \mu_1 > 0.$$

Observe from (58) that m_{dual}^M satisfies $m_{dual}^M(1-\widehat{\tau}^M(m_{dual}^M,\overline{w})(1+W(\Omega(m_{dual}^M,\overline{w})))) > \mu_1$. This concludes the proof of the profit comparisons between the dual channel and direct channel strategies.

Proofs of Section 6

We first provide a lemma that will be used in the proofs of the propositions.

Lemma 4. *The optimal margins, m^{M} and $\overline{w}$, for $\Pi^M(S^R, m^{M}, \overline{w})$ in* (43) *must satisfy $F_1 = 0$ and $F_2 = 0$ where*

$$F_1 = m^{M}\left[1 - \widehat{\tau}^{M}(S^R, m^{M}, \overline{w})\left[1 + W(\Omega(S^R, m^{M}, \overline{w}))\right]\right] - \mu_1\left[1 + \sum_{j\in S^R}\theta_j \widehat{q}_j^{R}(S^R, m^{M}, \overline{w}) W(\Omega(S^R, m^{M}, \overline{w}))\right],$$

and

$$F_2 = \sum_{j\in S^R}\left[\overline{w} - \mu_2\theta_j \widehat{q}_j^{R}(S^R, m^{M}, \overline{w})\right]\left[1 - \widehat{\tau}^{R}(S^R, m^{M}, \overline{w})\right] - \sum_{j\in S^R}\theta_j \widehat{q}_j^{R}(S^R, m^{M}, \overline{w})\mu_1[1 + W(\Omega(S^R, m^{M}, \overline{w}))] - m^{M}\widehat{\tau}^{M}(S^R, m^{M}, \overline{w}).$$

Proof of Lemma 4: Recall from the proof of Proposition 3 that any optimal price vector for the manufacturer, $\boldsymbol{p}^{M}$ and $\boldsymbol{w}$, must satisfy FOCs. Therefore, m^{M} and $\overline{w}$ will also satisfy FOCs. We next derive the functions F_1 and F_2 using as our starting point the FOCs of $\Pi^M(S^R, m^{M}, \overline{w})$ as defined in Lemma 2.

Recall the FOCs of $\Pi^M(S^R, m^{M}, \overline{w})$ with respect to m^{M} and $\overline{w}$ provided in (56) and (57). Notice from two expressions that the RHS of (56) and the LHS of (57) are the same. Hence, to derive the function F_1 we replace the LHS of (56) with the RHS of (57) to obtain that:

$$m^{M}\left[1 - \widehat{\tau}^{M}(S^R, m^{M}, \overline{w})\left[1 + W(\Omega(S^R, m^{M}, \overline{w}))\right]\right] - \mu_1\left[1 + \sum_{j\in S^R}\theta_j \widehat{q}_j^{R}(S^R, m^{M}, \overline{w}) W(\Omega(S^R, m^{M}, \overline{w}))\right] = 0.$$

Observe that the left-hand side of the expression above is the same as F_1 as defined in the lemma.

As for F_2, it follows directly from (57).

Proof of Proposition 7: Let m^{M*} and $\overline{w}^*$ be the margins that satisfy the conditions $F_1 = 0$ and $F_2 = 0$ (provided in Lemma 4) for a given assortment. This proof is done in two parts: In

Part 1, we show that the manufacturer's expected profit, $\Pi^M(S^R, m^{M*}, \overline{w}^*)$ in (43), increases in α_k for $k \in S^R$. In Part 2, we show that

$$\lim_{\alpha_z \to -\infty} \Pi^M(S^R \cup \{z\}, m^{M*}, \overline{w}^*) = \Pi^M(S^R, m^{M*}, \overline{w}^*).$$

Part 1: We note below the derivatives of $\Omega(S^R, m^{\scriptscriptstyle M}, \overline{w})$, $\widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}))$, and $\widehat{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w})$, all provided in Lemma 2 with respect to α_k.

$$\begin{aligned}
\frac{\partial \Omega(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \alpha_k} &= \Omega(S^R, m^{\scriptscriptstyle M}, \overline{w}) \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_1}, \\
\frac{\partial \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}))}{\partial \alpha_k} &= \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})[1 - \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})]^2 \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_1}, \\
\frac{\partial \widehat{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \alpha_k} &= \widehat{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w}) \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}) \\
&\qquad \times [1 - \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})]^2 \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_1}, \\
\frac{\partial \widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \alpha_k} &= \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_2} \left[1 - \widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})\right], k \in S^R, \\
\frac{\partial \widehat{q}_j^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \alpha_k} &= -\widehat{q}_j^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}) \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_2}, k \in S^R, k \neq j.
\end{aligned}$$

Using the derivatives above and given the fact that

$$\left.\frac{\partial \Pi^M(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial m^{\scriptscriptstyle M}}\right|_{m^M = m^{M*}} = 0 \text{ and } \left.\frac{\partial \Pi^M(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \overline{w}}\right|_{\overline{w} = \overline{w}^*} = 0,$$

we find that:

$$\begin{aligned}
\frac{\mathbf{d}\Pi^M(S^R, m^{M*}, \overline{w}^*)}{\mathbf{d}\alpha_k} &= \frac{\partial \Pi^M(S^R, m^{M*}, \overline{w}^*)}{\partial \alpha_k} \\
&= \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{M*}, \overline{w}^*)[1 - \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{M*}, \overline{w}^*)] \\
&\quad + \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{M*}, \overline{w}^*)}{\mu_1} \left[F_2 + \frac{\mu_1 \theta_k}{1 - \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{M*}, \overline{w}^*)}\right],
\end{aligned} \tag{63}$$

where F_2 is as defined in Lemma 4. Recall that the optimal margins for the manufacturer will satisfy the conditions $F_1 = 0$ and $F_2 = 0$; hence

$$\frac{\mathbf{d}\Pi^M(S^R, m^{M*}, \overline{w}^*)}{\mathbf{d}\alpha_k} > 0.$$

Part 2: For any given pair $\overline{w}$ and m^{M}, it can be shown that:

$$\begin{aligned}
&\lim_{\alpha_z\to-\infty} \widehat{\tau}^{M}(S^R\cup\{z\}, m^{M}, \overline{w}) = \widehat{\tau}^{M}(S^R, m^{M}, \overline{w}),\\
&\lim_{\alpha_z\to-\infty} \Omega(S^R\cup\{z\}, m^{M}, \overline{w}) = \Omega(S^R, m^{M}, \overline{w}),\\
&\lim_{\alpha_z\to-\infty} \widehat{\tau}^{R}(S^R\cup\{z\}, m^{M}, \overline{w}) = \widehat{\tau}^{R}(S^R, m^{M}, \overline{w}),\\
&\lim_{\alpha_z\to-\infty} \sum_{\substack{j\in S^R\\ j\neq z}} \widehat{q}_j^{R}(S^R\cup\{z\}, m^{M}, \overline{w}) = \sum_{\substack{j\in S^R\\ j\neq z}} \widehat{q}_j^{R}(S^R, m^{M}, \overline{w}),\\
&\lim_{\alpha_z\to-\infty} \widehat{q}_z^{R}(S^R\cup\{z\}, m^{M}, \overline{w}) = 0.
\end{aligned}$$

Using the above limits along with F_1 and F_2 as defined in Lemma 4, note that any pair of m^{M} and $\overline{w}$ that satisfy F_1 and F_2 for the assortment S^R will also satisfy F_1 and F_2 for the assortment $S^R\cup\{z\}$ when α_z goes to negative infinity. Furthermore, using the same limits and the expression for $\Pi^M(m^{M}, \overline{w})$ as defined in Lemma 2, it follows that:

$$\lim_{\alpha_z\to-\infty} \Pi^M(S^R\cup\{z\}, m^{M}, \overline{w}) = \Pi^M(S^R, m^{M}, \overline{w}).$$

Putting together the facts that the manufacturer's profit function is increasing in α_z, and that at a very small α_z the manufacturer's profit function is at least as good as the assortment that does not includes variant z, we conclude the proof.

Proof of Proposition 8: From Proposition 2 we know that the optimal effective margin for the retailer is $m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}) = \frac{\mu_1}{1-\tau^R(S^R,\boldsymbol{p}^M, m^R(S^R,\boldsymbol{p}^M,\boldsymbol{w}))}$. Hence we can write the profit function for the retailer given in (16) as:

$$\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}, m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})) = m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}) - \mu_1. \tag{64}$$

Observe from above that the right hand side is increasing in $m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})$. Hence, this proof will follow after showing that $m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})$ always increases with the addition of a new variant to the retailer's assortment.

It follows from (23) that:

$$\frac{m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})}{\mu_1} - 1 = W\left(\frac{\exp(-1)\left[\sum_{k\in S^R}\exp(\eta_k^{R}/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}\right). \tag{65}$$

Notice that the right-hand side of the expression above is increasing in $m^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{w})$ and the right-hand side depends only on problem parameters for the retailer's pricing problem. Hence, any change in parameters that will force the right-hand side to increase will in fact increase

$m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})$. Let z be an available variant that can be added to the retailer's assortment, i.e. $z \in S^M$, $z \notin S^R$, and recall that the Lambert-W function is increasing in its arguments. Observe that introducing this variant into the retailer's assortment will produce the following relationship,

$$\begin{aligned}
\frac{m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})}{\mu_1} - 1 &= W\left(\frac{\exp(-1)\left[\sum_{k\in S^R}\exp(\eta^{\scriptscriptstyle R}_k/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p^{\scriptscriptstyle M}_k)\right]^{\mu_2/\mu_1}}\right)\\
&< W\left(\frac{\exp(-1)\left[\sum_{k\in S^R}\exp(\eta^{\scriptscriptstyle R}_k/\mu_2) + \exp(\eta^{\scriptscriptstyle R}_z/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p^{\scriptscriptstyle M}_k)\right]^{\mu_2/\mu_1}}\right)\\
&= \frac{m^{\scriptscriptstyle R}(S^R\cup\{z\}, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})}{\mu_1} - 1.
\end{aligned}$$

Hence, $m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}) < m^{\scriptscriptstyle R}(S^R\cup\{z\}, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})$ which implies

$$\Pi^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})) < \Pi^R(S^R\cup\{z\}, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})).$$

Proof of Proposition 9: Observe that the manufacturer's profit under Scenario 1 will always be at least as high as the manufacturer's profit under Scenario 2 (because under Scenario 1 the manufacturer chooses the retailer's assortment while the retailer makes the same decision under Scenario 2). Now, recall that the manufacturer picks its prices before the retailer. Hence, even under Scenario 2 the manufacturer will pick the same variant that it would pick under Scenario 1 and will choose the same wholesale and direct channel prices that it would choose under Scenario 1. The manufacturer will make all other variants unattractive for the retailer, which could be achieved by charging a very high wholesale price. Hence, under Scenario 2, the retailer will offer the variant that the manufacturer will offer under Scenario 1.

Proof of Proposition 10: For this proof, we set up the simplified versions of the retailer's and manufacturer's pricing problems. Consider $S^M = \{1, 2\}$. Suppose the retailer offers variant k, i.e. $S^R = \{k\}$. We will prove this result by showing that the manufacturer's and retailer's profits, evaluated at the equilibrium wholesale and direct channel prices, are increasing in γ_k.

Let k be the only variant offered at the retailer, i.e. $S^R = \{k\}$. Since this is a special case of the more general model, the two conditions described in Lemma 4 continue to hold. For exposition purposes, we are going to make a few observations taking advantage of the fact that $S^R = \{k\}$.

Observation 1: Let

$$\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k) = \frac{\exp(-1)\left[\exp([\alpha_k - \gamma_k - w_k]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp\left(\alpha_0\right) + \left[\sum_{j \in S^M} v_j^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}. \tag{66}$$

From Proposition 3 we use the pricing property that $\overline{w} = (w_k - c_k - \mu_2)\theta_k$ to observe that $\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k) = \Omega(S^R, m^{\scriptscriptstyle M}, \overline{w})$, where $\Omega(S^R, m^{\scriptscriptstyle M}, \overline{w})$ is defined in Lemma 2. Using the same pricing property, we observe that:

$$\sum_{j \in S^R} \left[\overline{w} - \mu_2 \theta_j\right] \widehat{q}_j^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}) = (w_k - c_k)\theta_k.$$

Observation 2: Let

$$\widetilde{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, w_k) = \frac{W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k))}{1 + W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k))}, \tag{67}$$

and

$$\widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) = \frac{\left[\sum_{j \in S^M} v_j^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}{\left[\exp\left(\alpha_0\right) + \left[\sum_{j \in S^M} v_j^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}\right]\left[1 + W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k))\right]}. \tag{68}$$

Using the expressions for $\tau^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})$ and $\tau^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w})$ in Lemma 2 together with Observation 1, note that

$$\widetilde{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, w_k) = \tau^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}) \text{ and } \widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) = \tau^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w}).$$

Using Observations 1 and 2, we can now write the manufacturer's profit function in Lemma 2 as:

$$\widetilde{\Pi}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) = \widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) m^{\scriptscriptstyle M} + \widetilde{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, w_k)(w_k - c_k)\theta_k. \tag{69}$$

Furthermore, we can write the conditions $F_1 = 0$ and $F_2 = 0$, defined in Lemma 4 as:

$$\begin{aligned}\widetilde{F}_1 = {} & m^{\scriptscriptstyle M}\left[1 - \widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k)\left[1 + W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k))\right]\right] \\ & - \mu_1[1 + \theta_k W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k)] = 0.\end{aligned} \tag{70}$$

and

$$\begin{aligned}\widetilde{F}_2 = {} & -m^{\scriptscriptstyle M} \widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) \\ & + (w_k - c_k)\theta_k\left[1 - \widetilde{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, w_k)\right] - \frac{\mu_1 \theta_k}{1 - \widetilde{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, w_k)} = 0.\end{aligned} \tag{71}$$

Observation 3: It can be shown that the derivatives of $\widetilde{\tau}^{R}(S^R, m^M, w_k)$, given by (67), and $\widetilde{\tau}^{M}(S^R, m^M, w_k)$, given by (68), with respect to γ_k are:

$$\frac{\partial\widetilde{\tau}^{R}(S^R, m^M, w_k)}{\partial\gamma_k} = \frac{-\widetilde{\tau}^{R}(S^R, m^M, w_k)[1-\widetilde{\tau}^{R}(S^R, m^M, w_k)]^2}{\mu_1},$$
$$\frac{\partial\widetilde{\tau}^{M}(S^R, m^M, w_k)}{\partial\gamma_k} = \frac{\widetilde{\tau}^{M}(S^R, m^M, w_k)\widetilde{\tau}^{R}(S^R, m^M, w_k)[1-\widetilde{\tau}^{R}(S^R, m^M, w_k)]}{\mu_1}.$$

Let m^{M*} and w^* denote the margin and wholesale price in equilibrium. Given that m^{M*} and w^* satisfy FOCs of $\widetilde{\Pi}^M(S^R, m^M, w)$, we use the derivatives above find that:

$$\frac{\mathbf{d}\widetilde{\Pi}^M(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} = \frac{\partial\widetilde{\Pi}^M(S^R, m^{M*}, w^*)}{\partial\gamma_k} =$$

$$\frac{\widetilde{\tau}^{R}(S^R, m^{M*}, w^*)[1-\widetilde{\tau}^{R}(S^R, m^{M*}, w^*)]}{\mu_1}\begin{bmatrix} m^{M*}\widetilde{\tau}^{M}(S^R, m^{M*}, w^*) \\ -(w^*-c_k)\theta_k[1-\widetilde{\tau}^{R}(S^R, m^{M*}, w^*)] \end{bmatrix}.$$

Because m^{M*} and w^* satisfy (71), we have

$$(w^*-c_k)\theta_k[1-\widetilde{\tau}^{R}(S^R, m^{M*}, w^*)] > m^{M*}\widetilde{\tau}^{M}(S^R, m^{M*}, w^*).$$

Therefore:

$$\frac{\mathbf{d}\widetilde{\Pi}^{M}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} < 0.$$

Given that (i) co_k and cu_k play a role only in the function γ_k, (ii) γ_k is increasing in co_k and cu_k, and (iii) $\frac{\mathbf{d}\widetilde{\Pi}^{M}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} < 0$, we now conclude that $\widetilde{\Pi}^{M}(S^R, m^{M*}, w^*)$ is decreasing in co_k and cu_k.

Observation 4: Recall from (42) that $m^{R}(m^{M*}, \overline{w}^*) = \mu_1[1 + W(\Omega(m^{M*}, \overline{w}^*))]$. Given Observation 1, we can write

$$m^{R}(m^{M*}, w^*) = \mu_1[1 + W(\Omega(m^{M*}, w^*))].$$

Replacing $m^{R}(m^{M*}, w^*)$ with $\mu_1[1 + W(\Omega(m^{M*}, w^*))]$ in (64), we can write

$$\widetilde{\Pi}^R(S^R, m^{M*}, w^*) = W(\widetilde{\Omega}(S^R, m^{M*}, w^*))\mu_1 \tag{72}$$

Given that the function $W(\cdot)$ is increasing in its arguments and μ_1 is a problem parameter, observe that the expression for $\widetilde{\Pi}^R(S^R, m^{M*}, w^*)$ provided in (72) is strictly increasing in $\widetilde{\Omega}(S^R, m^{M*}, w^*)$. Then to show that $\widetilde{\Pi}^R(S^R, m^{M*}, w^*)$ is increasing in γ_k it suffices to show that:

$$\frac{\mathbf{d}\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} < 0.$$

Observation 5: Taking the derivative of $\widetilde{\Omega}(S^R, m^{M*}, w^*)$, given by (66), with respect to γ_k provides:

$$\frac{\mathbf{d}\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} = \frac{\partial\widetilde{\Omega}(S^R, m^{M}, w)}{\partial\gamma_k} + \left.\frac{\partial\widetilde{\Omega}(S^R, m^{M}, w)}{\partial m^{M}}\right|_{m^M=m^{M*}} \frac{\mathbf{d}m^{M*}}{\mathbf{d}\gamma_k}$$
$$+ \left.\frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial w^*}\right|_{w=w^*} \frac{\mathbf{d}w}{\mathbf{d}\gamma_k}.$$

We will observe that the above expression is negative after showing that:

$$\frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial\gamma_k} + \frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial w_k}\frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k} < 0 \text{ and } \frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial m^{M*}}\frac{\mathbf{d}m^{M}}{\mathbf{d}\gamma_k} < 0.$$

Recall the two conditions that m^{M*} and w^* must satisfy for the manufacturer's problem, i.e. $\widetilde{F}_1 = 0$ and $\widetilde{F}_2 = 0$. Using implicit differentiation we have that:

$$\frac{\mathbf{d}m^{M*}}{\mathbf{d}\gamma_k} = \frac{-\frac{\partial\widetilde{F}_1}{\partial\gamma_k}\frac{\partial\widetilde{F}_2}{\partial w^*} + \frac{\partial\widetilde{F}_2}{\partial\gamma_k}\frac{\partial\widetilde{F}_1}{\partial w_k}}{\frac{\partial\widetilde{F}_1}{\partial m^{M*}}\frac{\partial\widetilde{F}_2}{\partial w^*} - \frac{\partial\widetilde{F}_2}{\partial m^{M}}\frac{\partial\widetilde{F}_1}{\partial w^*}}, \text{ and} \tag{73}$$

$$\frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k} = \frac{-\frac{\partial\widetilde{F}_2}{\partial\gamma_k}\frac{\partial\widetilde{F}_1}{\partial m^{M*}} + \frac{\partial\widetilde{F}_1}{\partial\gamma_k}\frac{\partial\widetilde{F}_2}{\partial m^{M*}}}{\frac{\partial\widetilde{F}_1}{\partial m^{M*}}\frac{\partial\widetilde{F}_2}{\partial w^*} - \frac{\partial\widetilde{F}_2}{\partial m^{M*}}\frac{\partial\widetilde{F}_1}{\partial w^*}}. \tag{74}$$

It can be shown that:

$$\frac{\partial\widetilde{\Omega}(S^R, m^{M}, w)}{\partial\gamma_k} = \frac{-\widetilde{\Omega}(S^R, m^{M}, w)}{\mu_1},$$
$$\frac{\partial\widetilde{\tau}^{R}(S^R, m^{M}, w)}{\partial\gamma_k} = \frac{-\widetilde{\tau}^{R}(S^R, m^{M}, w)[1-\widetilde{\tau}^{R}(S^R, m^{M}, w)]^2}{\mu_1},$$
$$\frac{\partial\widetilde{\tau}^{M}(S^R, m^{M}, w)}{\partial\gamma_k} = \frac{-\widetilde{\tau}^{M}(S^R, m^{M}, w)\widetilde{\tau}^{R}(S^R, m^{M}, w)[1-\widetilde{\tau}^{R}(S^R, m^{M}, w)]}{\mu_1},$$
$$\frac{\partial\widetilde{F}_1}{\partial\gamma_k} = \theta_k\widetilde{\tau}^{R}(S^R, m^{M}, w),$$
$$\frac{\partial\widetilde{F}_2}{\partial\gamma_k} = 2\theta_k\widetilde{\tau}^{R}(S^R, m^{M}, w) + \frac{\widetilde{\tau}^{R}(S^R, m^{M}, w)[1-\widetilde{\tau}^{R}(S^R, m^{M}, w)]}{\mu_1}\widetilde{F}_2,$$
$$\frac{\partial\widetilde{F}_1}{\partial w} = \theta_k\widetilde{\tau}^{R}(S^R, m^{M}, w),$$
$$\frac{\partial\widetilde{F}_2}{\partial w} = \theta_k[1+\widetilde{\tau}^{R}(S^R, m^{M}, w)] + \frac{\widetilde{\tau}^{R}(S^R, m^{M}, w)[1-\widetilde{\tau}^{R}(S^R, m^{M}, w)]}{\mu_1}\widetilde{F}_2,$$
$$\frac{\partial\widetilde{F}_1}{\partial m^{M}} = \widetilde{F}_1\widetilde{\tau}^{M}(S^R, m^{M}, w)[1+W(\widetilde{\Omega}(S^R, m^{M}, w)]$$
$$+ 1 + \widetilde{\tau}^{M}(S^R, m^{M}, w)\theta_k[W(\widetilde{\Omega}(S^R, m^{M}, w)]^2,$$
$$\frac{\partial\widetilde{F}_2}{\partial m^{M}} = -\theta_k\widetilde{\tau}^{M}(S^R, m^{M}, w)W(\widetilde{\Omega}(m^{M}, w)) + \frac{\widetilde{\tau}^{M}(S^R, m^{M}, w)}{\mu_1}$$
$$\times\left[\widetilde{F}_1 - \widetilde{F}_2\widetilde{\tau}^{R}(S^R, m^{M}, w)\right].$$

Using the derivatives above it will follow that $\frac{\mathbf{d}m^{M*}}{\mathbf{d}\gamma_k}$, given by (73), is negative. Observe now that:

$$\frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial\gamma_k} + \frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial w}\frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k} = \frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial\gamma_k}\left[1 + \frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k}\right] < 0.$$

This observation follows after noting that that $1 + \frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k} > 0$, and $\frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial\gamma_k} < 0$, by using the derivatives of the conditions $\widetilde{F}_1$ and $\widetilde{F}_2$ listed previously.

Using Observations 4 and 5, we conclude

$$\frac{\mathbf{d}\widetilde{\Pi}^{R}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} < 0.$$

Since γ_k is increasing in co_k and cu_k we conclude that $\widetilde{\Pi}^{R}(S^R, m^{M*}, w^*)$ is decreasing in co_k and cu_k.

Proof of Proposition 11: In this proposition we assume all variants are the same (which implies θ_k is the same for all $k \in S^R$). Hence, we have that w_k is the same for all $k \in S^R$. Given that all variants are the same, it is no longer the composition of the retailer's assortment, but the size of the retailer's assortment that influences profits. Let M denote the number of variants carried by the manufacturer and let N denote the number of variants carried by the retailer. For notational convenience let $w = w_k$, $\alpha = \alpha_k$, $c = c_k$, $\gamma = \gamma_k$ and $\theta = \theta_k$.

Let $\overline{\Pi}^M(N, m^{M}, w)$ be the manufacturer's profit and $\overline{\Pi}^R(N, m^{M}, w)$ be the retailer's profit for this simplified problem. The proof of this proposition will follow after showing that:

$$\frac{\mathbf{d}\overline{\Pi}^M(N, m^{M*}, w^*)}{\mathbf{d}N} > \frac{\mathbf{d}\overline{\Pi}^R(N, m^{M*}, w^*)}{\mathbf{d}N},$$

Observation 1: Let

$$\overline{\Omega}(N, m^{M}, w) = \frac{\exp(-1)\,[N\exp([\alpha - \gamma - w]/\mu_2)]^{\mu_2/\mu_1}}{\exp(\alpha_0) + [M\exp([\alpha - c - m^{M}]/\mu_2)]^{\mu_2/\mu_1}}. \quad (75)$$

Given that all variants are the same for the purposes of this proposition, then observe that $\overline{\Omega}(N, m^{M}, w) = \Omega(S^R, m^{M}, \overline{w})$, where $\Omega(S^R, m^{M}, \overline{w})$ is defined in Lemma 2.

Observation 2: Let

$$\overline{\tau}^{R}(N, m^{M}, w) = \frac{W(\overline{\Omega}(N, m^{M}, w))}{1 + W(\overline{\Omega}(N, m^{M}, w))}, \quad (76)$$

and

$$\overline{\tau}^{M}(N, m^{M}, w) = \frac{[M\exp([\alpha - c - m^{M}]/\mu_2)]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + [M\exp([\alpha - c - m^{M}]/\mu_2)]^{\mu_2/\mu_1}\right][1 + W(\overline{\Omega}(N, m^{M}, w))]}. \quad (77)$$

Using Observation 1, note that $\overline{\tau}^{R}(N, m^{M}, w) = \tau^{R}(S^{R}, m^{M}, \overline{w})$ and $\overline{\tau}^{M}(N, m^{M}, w) = \tau^{M}(S^{R}, m^{M}, \overline{w})$ where $\tau^{R}(S^{R}, m^{M}, \overline{w})$ and $\tau^{M}(S^{R}, m^{M}, \overline{w})$ are defined in Lemma 2.

Observation 3: Let

$$\overline{\Pi}^{R}(N, m^{M}, w) = \mu_1 W(\overline{\Omega}(N, m^{M}, w)),$$

and

$$\Pi^{M}(N, m^{M}, w) = \overline{\tau}^{M}(N, m^{M}, w)m^{M} + \overline{\tau}^{R}(N, m^{M}, w)(w - c)\theta, \tag{78}$$

Using Observation 2, note that

$$\overline{\Pi}^{R}(N, m^{M}, w) = \Pi^{R}(S^{R}, m^{M}, \overline{w}) \text{ and } \overline{\Pi}^{M}(N, m^{M}, w) = \Pi^{M}(S^{R}, m^{M}, \overline{w}).$$

Given that the problem in this proposition is a simplification of the general problem, the conditions in Lemma 4 continue to hold. Using Observations 1 through 3, conditions F_1 and F_2 in Lemma 4 can now be simplified to $\overline{F}_1$ and $\overline{F}_2$ as follows:

$$\overline{F}_1 = m^{M}\left[1 - \overline{\tau}^{M}(N, m^{M}, w)\left[1 + W(\overline{\Omega}(N, m^{M}, w))\right]\right] - \mu_1[1 + \theta W(\overline{\Omega}(N, m^{M}, w))],$$

and

$$\overline{F}_2 = \frac{\mu_1 \theta}{1 - \overline{\tau}^{R}(N, m^{M}, w)} + m^{M}\overline{\tau}^{M}(N, m^{M}, w) - (w - c)\theta[1 - \overline{\tau}^{R}(N, m^{M}, w)].$$

The conditions above will be used later to find the derivatives of the retailer's and manufacturer's profit functions.

We next provide some partial derivatives that will help with the derivations of the profit functions. The derivatives for $\overline{\Omega}(N, m^{M}, w)$ in (75), $\overline{\tau^{R}}(N, m^{M}, w)$ in (76), and $\overline{\tau^{M}}(N, m^{M}, w)$

in (77) with respect to N, $m^{\scriptscriptstyle M}$ and w_k are:

$$
\begin{aligned}
\frac{\partial \overline{\Omega}(N,m^{\scriptscriptstyle M},w_k)}{\partial N} &= \frac{\mu_2}{\mu_1 N}\overline{\Omega}(N,m^{\scriptscriptstyle M},w_k),\\
\frac{\partial \overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)}{\partial N} &= \frac{\mu_2}{\mu_1 N}\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)[1-\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)]^2,\\
\frac{\partial \overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)}{\partial N} &= -\frac{\mu_2}{\mu_1 N}\overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)[1-\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)],\\
\frac{\partial \overline{F}_1}{\partial N} &= -\frac{\mu_2}{N}\theta\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k),\\
\frac{\partial \overline{F}_2}{\partial N} &= 2\frac{\mu_2}{N}\theta\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k),\\
\frac{\partial \overline{\Omega}(N,m^{\scriptscriptstyle M},w_k)}{\partial m^{\scriptscriptstyle M}} &= \frac{\overline{\Omega}(N,m^{\scriptscriptstyle M},w_k)}{\mu_1}\overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)[1+W(\overline{\Omega}(N,m^{\scriptscriptstyle M},w_k))],\\
\frac{\partial \overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)}{\partial m^{\scriptscriptstyle M}} &= \frac{[1-\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)]}{\mu_1}\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)\overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k),\\
\frac{\partial \overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)}{\partial m^{\scriptscriptstyle M}} &= -\frac{\overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)}{\mu_1}\left[1+\overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)[\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)]\right.\\
&\qquad \left. -1-W(\overline{\Omega}(N,m^{\scriptscriptstyle M},w_k))\right],\\
\frac{\partial \overline{F}_1}{\partial m^{\scriptscriptstyle M}} &= 1+\overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)\theta[W(\overline{\Omega}(N,m^{\scriptscriptstyle M},w_k))]^2,\\
\frac{\partial \overline{F}_2}{\partial m^{\scriptscriptstyle M}} &= \overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)\theta W(\overline{\Omega}(N,m^{\scriptscriptstyle M},w_k)),\\
\frac{\partial \overline{\Omega}(N,m^{\scriptscriptstyle M},w_k)}{\partial w_k} &= -\frac{\overline{\Omega}(N,m^{\scriptscriptstyle M},w_k)}{\mu_1},\\
\frac{\partial \overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)}{\partial w_k} &= -\frac{\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)}{\mu_1}[1-\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)]^2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)}{\partial w_k} &= \frac{\overline{\tau}^{\scriptscriptstyle M}(N,m^{\scriptscriptstyle M},w_k)\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)}{\mu_1}[1-\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)],\\
\frac{\partial \overline{F}_1}{\partial w_k} &= \theta\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k),\\
\frac{\partial \overline{F}_2}{\partial w_k} &= -\theta[1+\overline{\tau}^{\scriptscriptstyle R}(N,m^{\scriptscriptstyle M},w_k)].
\end{aligned}
$$

Using implicit differentiation we find the following derivatives with respect to N for the manufacturer and retailer:

$$\frac{\mathbf{d}\Pi^M(N, m^{M*}, w^*)}{\mathbf{d}N} = \frac{\mu_2 \overline{\tau}^{R}(N, m^{M*}, w^*)\theta}{N}, \tag{79}$$

$$\frac{\mathbf{d}\Pi^R(N, m^{M*}, w^*)}{\mathbf{d}N} = \frac{\mu_2 \overline{\tau}^{R}(N, m^{M*}, w^*)\theta}{N} \times \frac{\overline{\tau}^{M}(N, m^{M*}, w^*)\theta \left[1 + W(\overline{\Omega}(N, m^{M*}, w^*))\right] + 1/W(\overline{\Omega}(N, m^{M*}, w^*))}{\begin{bmatrix} 2\overline{\tau}^{M}(N, m^{M*}, w^*)\theta W(\overline{\Omega}(N, m^{M*}, w^*)) + 1/W(\overline{\Omega}(N, m^{M*}, w^*)) \\ + 2[1 + \overline{\tau}^{M}(N, m^{M*}, w^*)\theta[W(\overline{\Omega}(N, m^{M*}, w^*))]^2] \end{bmatrix}}. \tag{80}$$

Observe that (80) is smaller than (79) when

$$\frac{\overline{\tau}^{M}(N, m^{M*}, w^*)\theta \left[1 + W(\overline{\Omega}(N, m^{M*}, w^*))\right] + 1/W(\overline{\Omega}(N, m^{M*}, w^*))}{\begin{bmatrix} 2\overline{\tau}^{M}(N, m^{M*}, w^*)\theta W(\overline{\Omega}(N, m^{M*}, w^*)) + 1/W(\overline{\Omega}(N, m^{M*}, w^*)) \\ + 2[1 + \overline{\tau}^{M}(N, m^{M*}, w^*)\theta[W(\overline{\Omega}(N, m^{M*}, w^*))]^2] \end{bmatrix}} < 1. \tag{81}$$

One can show that the above condition holds when

$$\overline{\tau}^{M}(N, m^{M*}, w^*) \left[1 - W(\overline{\Omega}(N, m^{M*}, w^*)) - 2W(\overline{\Omega}(N, m^{M*}, w^*))^2\right] < \frac{2}{\theta}.$$

Hence, we conclude that when $\theta < 2$ (80) is smaller than (79).