Pricing European and American options of real estate index

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Abstract
In this paper, we modeled the real estate index options by no-arbitrage approach and obtained the partial differential equations (PDEs) of the real estate index options. Then we proposed a novel radial basis function (RBF) to solve the PDEs.

Keywords: Real Estate Index Options, No-arbitrage Approach, Radial Basis Functions

INTRODUCTION

Real estate derivatives market has emerged more than 20 years. However, in the beginning, little attention was paid to this market. Although the real estate derivatives market has displayed a rapid development for many years, the trading volume and liquidity are not comparable to financial derivatives because of investors' low acceptance of the real estate derivatives as new hedging instruments which can be partly attributed to the lack of reliable pricing models. In recent years, the pricing of real estate derivatives has attracted more and more attention. Buttimer et al. (1997) employed a bivariate binominal model to price the total return swap contingent on a real estate index and interest rate. They found a positive but negligible swap spread price. Bjork and Clapham (2002) revised the model proposed by Buttimer et al. (1997) and proved that the theoretical price of the total return swap is equal to zero when using the no-arbitrage approach. However, all the no-arbitrage models above ignore the fact that the real estate index is non-tradable. To solve this pricing problem, Geltner and Fisher (2007) proposed an equilibrium model for the pricing of real estate forwards and total return swap contracts. Fabozzi et al. (2012) considered the econometric properties of real estate indices and the incompleteness of real estate market by using the real estate futures market to complete such market. Under the consumption that the market price of the real estate index risk is known, the closed-form solutions were obtained for futures, European options and total return swaps contingent on real estate index. In this paper, we modeled the real estate index options by no-arbitrage approach and obtained the PDEs of the real estate index options. Then we develop a novel approach for solving the PDEs.

The common numerical approaches for options pricing are the binomial and trinomial trees, the finite difference (Hull and White 1990, O'Sullivan and O'Sullivan 2011), the finite element and finite volume methods (Zvan et al. 2001, Tangman et al. 2008). In addition, many researchers also used the meshfree methods which are effective to solve PDEs. Point interpolation method (PIM) is the most commonly used meshfree method and has been achieved remarkable progress in recent years. Polynomials basis function (PBF) is one of the earliest interpolation schemes. But, the method has a problem that polynomial basis possibly cause singularity (Liu and Gu 2005). After radial basis functions (RBFs) proposed, the shortcoming of PBFs has been improved largely. In this paper, we combined thin plate splines (TPSs) and PBFs to construct the RBFs.

REAL ESTATE INDEX OPTION PROBLEM

Real estate indices exhibit a positive autocorrelation in the short term and a negative autocorrelation in the long term. Therefore, we employed a mean reverting stochastic model to measure the real estate index movement (Fabozzi et al. 2012).

$$dY_t = [\frac{d\psi_t}{dt} - \theta(Y_t - \psi_t)]dt + \sigma dW_t \quad (1)$$

Where $Y_t = \log(X_t)$, the underlying asset X_t is the real estate index, ψ_t is the long run mean trend of real estate indices in log scale, and θ is the mean-reversion speed parameter.

Let $V(Y_t, t)$ denote the option price, we have

$$dV(Y_t,t) = \left(\frac{\partial V(Y_t,t)}{\partial t} + \mu \frac{\partial V(Y_t,t)}{\partial Y_t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(Y_t,t)}{\partial Y_t^2} \right) dt + \sigma \frac{\partial V(Y_t,t)}{\partial Y_t} dW_t \quad (2)$$

Hence

$$\frac{dV}{V} = \mu_V(Y_t,t)\,dt + \sigma_V(Y_t,t)\,dW_t \quad (3)$$

Considering that we construct a riskless hedging portfolio π that contains two derivatives of different maturities T_1 and T_2. Let Δ denote the number of units the derivatives. The portfolio is

$$\pi = V_1 - \Delta V_2 \tag{4}$$

In a very short period of time dt , the value changes of the portfolio is

$$d\pi = \left[V_1 \mu_{V_1}\left(Y_t, t\right) - \Delta V_2 \mu_{V_2}\left(Y_t, t\right)\right] dt \tag{5}$$

Under no arbitrage conditions, we have

$$\mu_V\left(Y_t, t\right) - r = \lambda\left(Y_t, t\right) \sigma_V\left(Y_t, t\right) \tag{6}$$

where $\lambda\left(Y_t, t\right)$ is the market price of risk.

Finally, we obtained the following PDE.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V}{\partial X_t^2} + \left(\mu - \lambda\sigma + \frac{1}{2}\sigma^2\right) X_t \frac{\partial V}{\partial X_t} - rV = 0 \tag{7}$$

We define following real estate index options Operator, so that we can easily describe in the section of American options.

$$LV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V}{\partial X_t^2} + \left(\mu - \lambda\sigma + \frac{1}{2}\sigma^2\right) X_t \frac{\partial V}{\partial X_t} - rV \tag{8}$$

European Options

For the sake of simplicity, we restrict our attention to put options, because call option can be treated in perfect analogy. Consider a put option with maturity T and strike price E. The final condition is

$$V_p\left(X_T, T\right) = g_p\left(X_T, T\right) = \left(K - X\right)^+ \tag{9}$$

Boundary conditions are

$$\begin{aligned} &V_p(0,\mathrm{t}) = K\exp\left(-r\left(T-t\right)\right) \\ &\lim_{X\to\infty} V_p\left(X,t\right) = 0 \end{aligned} \tag{10}$$

American Options

We still consider a put option, the final condition is

$$V_p\left(X_T,T\right) = g_p\left(X_T,T\right) = \left(K-X\right)^+ \tag{11}$$

The American option price V is not less than g_p. Hence, the equation is

$$\begin{cases} LV \geq 0\ , \quad V \geq g_p \\ \left(LV\right)\left(V-g_p\right) = 0 \end{cases} \tag{12}$$

Boundary conditions are

$$\begin{aligned} &V_p(0,t) = K \\ &\lim_{X\to\infty} V_p\left(X,t\right) = 0 \end{aligned} \tag{13}$$

TPS-PBF APPROXIMATION

The TPSs do not involve any free shape parameter. In particular, we employ the very popular TPSs of second order, which are

$$R_i\left(x\right) = \left(x-x_i\right)^4 \log\left(\left|x-x_i\right|\right) \quad i = 1,2,\cdots,n \tag{14}$$

Combined with PBFs is

$$u\left(x\right) = R^T\left(x\right)a + P^T\left(x\right)b \tag{15}$$

Where a and b are the coefficient vector of radial basis R and polynomial basis P respectively.

The polynomial term has to satisfy an extra requirement that guarantees unique approximation of a function, which is the following constraint.

$$P^T a = 0$$

We have the following system of linear equations

$$\begin{bmatrix} R & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = G \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} U \\ 0 \end{bmatrix} \tag{16}$$

The matrix R is non-singular, therefore

$$\begin{bmatrix} a \\ b \end{bmatrix} = G^{-1} \begin{bmatrix} U \\ 0 \end{bmatrix} \tag{17}$$

Equation (15) can be re-written as

$$u(x) = \begin{bmatrix} R^T(x) & P^T(x) \end{bmatrix} G^{-1} \begin{bmatrix} U \\ 0 \end{bmatrix} = \Phi(x)\overline{U} \tag{18}$$

TPS-PBF for Options

Employing TPS-PBFs to approximate the real estate index option price V, we have

$$V(X,t) = \Phi(X)\overline{U} \tag{19}$$

In order to numerically handle the unboundedness of the X-domain, we use the following change to variables

$$S = 1 - \exp(-X/L) \tag{20}$$

Where L is a parameter, which can be determined by equation $1-\exp\left(-E/L\right)=0.7$.

Then equation (7) can be rewritten as

$$\frac{\partial V}{\partial t}+A\left(S\right)\frac{\partial^2 V}{\partial S^2}+B\left(S\right)\frac{\partial V}{\partial S}-rV=0 \tag{21}$$

The RBF proposed in this paper is independent with time and determined the underlying asset *S*. Therefore, the equation(21) is continuously differentiable on *S*, and the following equation can be obtained.

$$\Phi V^{'}=\left(-A\left(S\right)\Phi_{SS}-B\left(S\right)\Phi_S+r\Phi\right)V=DV \tag{22}$$

European Options

In this paper, we divided the time interval [0, T] into M points, therefore $V^k\left(S\right)=V\left(S,k\Delta t\right),k=1,2,\cdots,M$. Combined with forward and backward difference scheme discretize the time derivative, we obtain the weight implicit scheme.

$$\Phi V^{k+1}=\Phi V^k+\Delta tD\left[\theta V^{k+1}+\left(1-\theta\right)V^k\right] \tag{23}$$

Let $P_1=\left[\Phi-\theta\Delta tD\right]$ and $P_2=\left[\Phi+\left(1-\theta\right)\Delta tD\right]$, equation (23) can rewrite as:

$$P_1V^{k+1}=P_2V^k \tag{24}$$

Due to the non-smoothness of the options' payoff, the Crank-Nicolson scheme fails to achieve its usual second-order accuracy. Therefore, we use the implicit Euler scheme, which is unconditionally stable and allows us to smooth the discontinuities of the options' payoffs.

American Options

Pricing American option is complicated because we must face a free boundary problem. Bermudan options allow early exercises at a finite number of pre-specified exercise times which is similar to American options. By increasing the number of exercise times we see that the Bermudan put option pricing is closely related to the American put option pricing. So we approximate the price of the American option with the price of a Bermudan option (Khaliq, Voss et al. 2006, Lim, Lee et al. 2014). According to equations(11)(12)(13), we consider an option which can be exercised not on the whole time interval [0, *T*], only at the dates $t_1, t_2, \cdots, t_M$.

Now, we assume that in each time interval $\left(t_k, t_{k+1}\right), k = 1, 2, \cdots, M$ the relations (11)(12)(13) holds true. That is we consider the problems

$$\begin{cases} LV = 0 \\ V_p\left(0, t\right) = K, \ \lim_{X \to \infty} V_p\left(X, t\right) = 0 \end{cases} \tag{25}$$

which hold true in the time intervals $\left(t_k, t_{k+1}\right), k = 1, 2, \cdots, M$. So the third relations of (12) is automatically satisfied in every time interval. The second relation of (12) is constraint only at times $t_1, t_2, \cdots, t_M$

$$V\left(X, t_k\right) = \max\left(\lim_{t \to t_k^+} V\left(t, X\right), g_p\left(X\right)\right), \ k = 1, 2, \cdots, M \tag{26}$$

Therefore, the solution of the American options is that

$$\begin{cases} P_1 V^{k+1} = P_2 V^k \\ V^k = \max\left(V^k, V^M\right) \end{cases} \tag{27}$$

where V^M is the option prices of maturity.

NUMERICAL RESULTS

Let $V_{RPBPI}\left(X_i, 0\right)$ denote the prices of using point interpolation method, $V\left(X_i, 0\right)$ imply the real prices of relative options. We defined the following error formulas

$$MaxError = \max_{i=1,2,\cdots,n} \left| V_{RPBPI}\left(X_i,0\right) - V\left(X_i,0\right) \right| \tag{28}$$

The mean square norm is

$$RmsError = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\left(V_{RPBPI}\left(X_i,0\right) - V\left(X_i,0\right)\right)^2} \tag{29}$$

European Options

Table 1 give the numerical results of put options when the strike price is 1800 and time to maturity is 1 year, risk-free rate is 0.04. The left part of Table 2 is the result by only TPS basis functions, and the right is obtained by the scheme of TPS-PBF. N is the number of scatter nodes in the domain, and Cond is the condition numbers of System matrix. As we can see, the option price can be computed with an error of order 10e-2 in the maximum norm and 10e-2 in the mean square norm. So the levels of accuracy is very high in this paper. Some researchers may suspect that the accuracy is not precise compared with stock options, which is understandable because they forgot the strike price is 1800 rather than 10 (Wilmott et al. 1995, Hon 2002).

Table 1 Put option prices at different basis functions

	RBPF			RBPF&PBPF		
N	**RmsError**	**MaxError**	**Cond/G**	**RmsError**	**MaxError**	**Cond/G**
50	1.7741E-02	5.4920E-01	3.0547E+08	1.1902E-02	2.6035E-01	7.1575E+09
75	6.2935E-03	2.8950E-01	2.5354E+09	4.6015E-03	1.7662E-01	6.0535E+10
100	3.4699E-03	1.9527E-01	1.1128E+10	2.8463E-03	1.3726E-01	2.6888E+11
125	2.4002E-03	1.4825E-01	3.4825E+10	2.1322E-03	1.1361E-01	8.4659E+11
150	1.9059E-03	1.2034E-01	8.8030E+10	1.7784E-03	9.7544E-02	2.1505E+12
200	1.4458E-03	8.8427E-02	3.7845E+11	1.4084E-03	7.6763E-02	9.2978E+12
250	1.2321E-03	7.1015E-02	1.1687E+12	1.2223E-03	6.3786E-02	2.8812E+13
300	1.1288E-03	5.8820E-02	2.9312E+12	1.0919E-03	5.4450E-02	7.2426E+13

It's obviously that, with the increasing of scatter nodes, the results become more and more precise. But with the increasing of scatter nodes the condition numbers of system matrix increase fast, this could cause the poor precision. Compared with only TPS basis functions, the novel method can effectively improve the results accuracy.

American Options

Let us considering American option which strike price is 1500 and time to maturity is 1 year, the results shown in table 2. We can see the same properties with European options, that with the number of scatter nodes increase the price tends to a stable value.

Table 2 The results of American option

American Put Options					
N	1450	1475	1500	1525	1550
100	75.9123	63.3945	52.0709	43.4651	35.5482
125	75.5831	63.2658	52.5959	43.3723	35.4823
150	75.5922	62.9376	52.1276	43.1181	35.4978
175	75.7149	63.1323	52.3914	43.2718	35.5705
200	75.6216	62.9817	52.1525	43.1305	35.4760
225	75.6087	62.9762	52.3057	43.1494	35.4586
250	75.4967	62.9867	52.1631	43.1441	35.3946
275	75.5347	62.9614	52.2637	43.1216	35.4541
300	75.5386	62.9358	52.1700	43.1063	35.4352

CONCLUSIONS

In developed countries, the real estate property have reached 30% to 40% of total market assets. However, the risk management tools available for hedging real-estate risk are very much in their infancy and have problems ranging from illiquidity of trading to lack of theoretical development in terms of modelling. In this paper, we advocated a suitable framework for pricing real estate options and proposed a novel method to solve the PDEs of European and American options.

The RBPI approach developed in this paper offers several advantages over the more conventional RBF approximation. First, the scheme combined TPSs and PBFs, which improves the accuracy and efficiency of the result compared with the pure RBFs. Second, we replace the original domain with a finite one, and don't introduce unknown finite boundaries and prescribe artificial conditions as in the previous methods. That is our innovation of variables change. Finally, we employed a local mesh refinement strategy, which allows us to easily and effectively handle the non-smoothness of the options' payoff. The results shown that the option prices can be computed with an error of order 10e-2 in the maximum norm and 10e-2 in the mean square norm, which is very precise respect to the options of large strike price.

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