# The integration of online and offline channels

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**Abstract**
We study a periodic review order-up-to policy for a two-echelon dual-channel system. The upper echelon satisfies online demand and replenishes the lower echelon while the lower echelon fulfills walk-in demand. In case of shortage, demand shift is considered. We evaluate the cost and characterize the optimal policy analytically and numerically.

**Keywords:** Two-echelon, Periodic reviews, Dual-channel

## Introduction

In a two-echelon dual-channel system, a company who sells a single product has two sales channels. One is direct shipping from the warehouse to a digital consumer and the other is satisfying a walk-in consumer via the on-hand inventory at a retail store where the store is replenished from the warehouse. Figure 1 depicts such a system.

There are some but not many literatures on inventory management for two-echelon dual-channel systems. Chiang and Monahan (2005) consider a one-for-one inventory control policy such that each fulfilled demand triggers an order at the upper stream. Numerical experiments demonstrate that the dual-channel strategy outperforms a single-channel one in terms of inventory holding and lost sales costs in most cases. Takahashi et al. (2011) propose an (s, S) policy for a system with setup costs. Production at the warehouse (resp. Delivery to the store) starts when the warehouse (resp. store) inventory drops below the minimum and stops when it reaches the maximum. They compare the proposed policy with the one-for-one and show that the proposed policy reduces both the number of setups and the total cost. Mahar et al. (2009) propose two dynamic assignment policies on how to assign an online order to one of the e-fulfillment locations in a real-time manner. The proposed policies lead to cost savings while compared to the optimal static policy.

In this paper, we study a two-echelon dual-channel system that adopts periodic review and consider how to allocate the total system stock to the warehouse and the store, so as to minimize system inventory costs (= holding + backordering cost). Both a static policy (the store order-up-to level at each review epoch is decided before time 0) and a dynamic policy (the store order-up-to level at the current review epoch is specified after observing past system demand) are investigated. Our contribution is twofold. First, under some circumstance, we find that past allocation decisions do not afflict future system inventory costs. Second, the optimal dynamic policy is superior to a static policy in all cases.

The remaining sections are organized as the following. We first specify assumptions and notations. After that, we conduct analysis on the two inventory control policies, i.e., a static policy and a dynamic one. Thereafter, numerical results are shown to illustrate the effectiveness of the dynamic policy. We summarize key findings in the end.

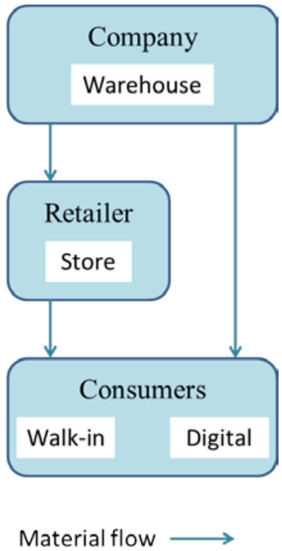


*Figure 1 – A two-echelon dual-channel supply chain*

# Model

## Assumptions

At time $t = mT_w,\ m \in \mathbb{N}$, the warehouse orders from an external party and raises the system inventory position IP(= on-hand inventory at the warehouse and the store + in-transit inventory to the warehouse and the store – consumer backorders at the warehouse and the store) to $S_0$. All of the ordered units arrive at the warehouse after the fixed lead time $l_w$. At each review epoch $t = l_w + nT_s,\ n \in \mathbb{N}$, the retailer places an order to the warehouse to raise store nominal inventory position NIPs(= on-hand inventory at the store + on-order inventory to the store – consumer backorders at the store) to $S_{s,n}$. If the warehouse has sufficient stock, it ships out all ordered units to the store; otherwise, it delivers all on-hand stock and backorders the remaining unsatisfied amount. This replenishment arrives at the store after the fixed lead time $l_s$. Moreover, at a store review epoch, if there are online backorders at the warehouse, we allow ship-from-store, i.e., delivering an online order by using store inventories. Walk-in and online demands are two independent Poisson processes with rate $\lambda_s$ and $\lambda_w$, respectively. A walk-in demand is satisfied by on-hand stock at the store while an online order is delivered directly from the warehouse. In case of no on-hand stock at the location, a consumer demand is backordered at the originating location with the backordering cost rate $b$ until its delivery starts (for an online order) or a unit becomes available at the store (for a walk-in order). Each unit kept at the store (resp. warehouse) is subject to a holding cost rate $h_s,\ h_s < b$ (resp. $h_w,\ h_w < b$) and $h_w < h_s$. We assume that $S_0$ is pre-determined and aim to determine the optimal $S_{s,n}$ to minimize system inventory costs. As the process restarts itself at time $t = mT_w,\ m \in \mathbb{N}$, we consider system inventory costs over the first $T_w$.

**Notations**

Table 1 defines other notations to be used in the next sections.

*Table 1 – Notations*

| Notations | Definitions |
|---|---|
| $C_n(IL, S_s)$ | Warehouse inventory costs in $[l_w + nT_s, l_w + (n+1)T_s)$ + store inventory costs in $[l_w + l_s + nT_s, l_w + l_s + (n+1)T_s)$, when system inventory level is $IL$ and store order-up-to level is $S_s$ at $l_w + nT_s$ |
| $C_{s,n}(IP_s)$ | Store inventory costs over $[l_w + l_s + nT_s, l_w + l_s + (n+1)T_s)$, when store inventory position at time $l_w + nT_s$ is $IP_s$ |
| $C_{w,n}(IL_w)$ | Warehouse inventory costs over $[l_w + nT_s, l_w + (n+1)T_s)$, when warehouse inventory level at time $l_w + nT_s$ is $IL_w$ |
| $D[t_1, t_2)$ (resp. $D_s, D_w$) | Total (resp. walk-in, online) consumer demand in the time interval $[t_1, t_2)$ |
| $G(d, \lambda)$ | Poisson cdf with mean $\lambda$ at $d$ |
| $n$ | An integer in $[0, T_w/T_s)$ |
| $S^{\bigstar}_{s,n}(IL)$ | The optimal store order-up-to level when system inventory level is $IL$ at time $l_w + nT_s$ in the dynamic environment |
| $S^*_{s,n}$ | The optimal store order-up-to level at $l_w + nT_s$ in the static environment |
| Store inventory position | = in-transit inventory + on-hand inventory – consumer backorders at the store |
| System inventory level | = on-hand inventory at the warehouse and the store + in-transit inventory to the store – consumer backorders at the warehouse and the store |
| Warehouse inventory level | = on-hand inventory – consumer backorders at the warehouse |
| $x^-$ | $= \max\{-x, 0\}$ |
| $x^+$ | $= \max\{x, 0\}$ |

## Analysis

In this section, we first prove that the minimization problem over $T_w$ is equivalent to $T_w/T_s$ independent minimization problems over $T_s$. Then we consider a static policy where the optimal store order-up-levels are determined before time 0. After that, we propose a dynamic policy which announces the optimal store order-up-to level after observing the current system inventory level.

*Proposition 1. $C_n$ is independent of $S_{s,i},\ i = 0, ..., n-1$.*

*Proof*. Given $D[0, l_w + nT_s) = d$, system inventory level at $l_w + nT_s$ is $S_0 - d$. From the assumptions we note that

$$C_n(S_0 - d, S_{s,n}) = C_{w,n}\big(\max\{0,\ S_0 - d - S_{s,n}\}\big) + C_{s,n}\big(\min\{S_0 - d,\ S_{s,n}\}\big) \qquad (1)$$

where

$$C_{w,n}(IL_w) = \mathbb{E}\Big[\int_0^{T_s} h_w\big[IL_w - D_w[t_0, t_0+t)\big]^+ + b\big[IL_w - D_w[t_0,t_0+t)\big]^-\ dt\Big] \tag{2}$$

$$C_{s,n}(IP_s) = \mathbb{E}\Big[\int_{l_s}^{l_s+T_s} h_s\big[IP_s - D_s[t_0, t_0+t)\big]^+ + b\big[IP_s - D_s[t_0,t_0+t)\big]^-\ dt\Big] \tag{3}$$

with $t_0 = l_w + nT_s$. All of the three equations (1), (2) and (3) are not related to $S_{s,i}$. Q.E.D.

In other words, past stock allocations do not affect future system inventory costs. Intuitively, it is because at a store review epoch, we are reallocating the net system stock which only depends on the past demand realization.

Remark. Note that in writing $C_n(S_0 - d, S_{s,n})$ as Equation (1), we are using two assumptions. One is that the backordering cost rate at the warehouse is the same as that at the retail store, so we can count the warehouse inventory level as non-negative and all warehouse online backorders left over from the previous $T_s$ period into store backorders. The other assumption used is that demand at one location is satisfied as long as there is on-hand inventory in this location, so that we have the integrands in Equations (2) and (3). However, when the two backordering cost rates are different, simply viewing warehouse online backorders as part of store backorders does not work; moreover, the second assumption needs to be re-justified as reserving a unit at the location with a low backordering cost rate for a demand at the other location seems reasonable.

**A Static Policy**

A static policy specifies $S_{s,n}^*$ before time 0. By Proposition 1, $S_{s,n}^*$ solves the following minimization problem,

$$\begin{aligned} \min \quad & \mathbb{E}\big[C_n(S_0 - D[0, l_w + nT_s),\ S_{s,n})\big], \\ \text{s.t.} \quad & 0 \le S_{s,n} \le S_0, \end{aligned} \tag{4}$$

where the expectation is taken over $D[0, l_w + nT_s)$.

We justify the constraint in Equation (4) as follows. A negative $S_{s,n}$ means that there is at least one backorders at the store. By increasing $S_{s,n}$ to 0, we are cutting down system backordering costs; meanwhile, the holding cost at the warehouse is reduced. On the other hand, Equation (1) implies that raising NIPs to more than $S_0$ does not change $\mathbb{E}\big[C_n\big]$ from $\mathbb{E}\big[C_n(S_0 - D[0, l_w + nT_s),\ S_0)\big]$.

**A Dynamic Policy**

We propose a dynamic policy to allocate net system stock based on the current system inventory level. Specifically, after observing $D[0, l_w + nT_s)$ to be $d$, we are to solve the following problem,

$$\begin{aligned} \min \quad & C_n(S_0 - d,\ S_{s,n}), \\ \text{s.t.} \quad & 0 \le S_{s,n} \le S_0 - d, \end{aligned} \tag{5}$$

where the justification for the constraint is similar to the one for Equation (4).

*Proposition 2.* $C_n(S_0 - d, S_{s,n})$ *is convex in* $S_{s,n},\ 0 \le S_{s,n} \le S_0 - d$.

*Proof.* For $\forall S_{s,n} \in [0, S_0 - d - 1]$, the first order difference of $C_n$ in $S_{s,n}$ is

$$\begin{aligned}\Delta C_n(S_{s,n}; S_0 - d) = C_n(S_0 - d, S_{s,n} + 1) - C_n(S_0 - d, S_{s,n}) \\ = \int_0^{T_s} -(h_w + b)G(S_0 - d - S_{s,n} - 1, \lambda_w t)\ dt \\ + \int_{l_s}^{l_s+T_s} (h_s + b)G(S_{s,n}, \lambda_s t)\ dt.\end{aligned} \tag{6}$$

It follows from Equation (6) that $\Delta C_n(S_{s,n}; S_0 - d)$ is increasing in $S_{s,n}$ and thus Proposition 2 holds. Q.E.D.

*Proposition 3. In terms of system inventory costs, the optimal dynamic policy is superior to the optimal static policy.*

*Proof.* $\mathbb{E}\big[C_n\big]$ while using the optimal dynamic policy is $\mathbb{E}\big[C_n(IL_n, S^{\bigstar}_{s,n}(IL_n))\big]$ and it is $\mathbb{E}\big[C_n(IL_n, S^{*}_{s,n})\big]$ if the optimal static policy is employed. Let $x$ be a realization of $IL_n$. Since $S^{\bigstar}_{s,n}(x) = \arg\min_{S_{s,n}} C_n(x, S_{s,n})$, we have

$$C_n(x, S^{\bigstar}_{s,n}(x)) \le C_n(x, S^{*}_{s,n}). \tag{7}$$

As Equation (7) holds for any $x$, we have

$$\mathbb{E}\big[C_n(IL_n, S^{\bigstar}_{s,n}(IL_n))\big] \le \mathbb{E}\big[C_n(IL_n, S^{*}_{s,n})\big], \tag{8}$$

and this completes the proof. Q.E.D.

We note that a dynamic policy requires the knowledge on the current system inventory level. Hence, it is applicable to a system whose real-time inventory level can be easily obtained.

Remark. In both static and dynamic policies, we assume that at a store review epoch store inventories can be used to satisfy online backorders at the warehouse and call it ship-from-store. Currently, the practice of ship-from-store is implemented by some companies including Gap Inc. (Gap Inc., 2013).

## Numerical Results

In this section, we conducted two sets of numerical experiments to show the effectiveness of our dynamic policy. An inventory costs saving from using the optimal dynamic policy (=(optimal static costs – optimal dynamic costs)/ optimal static costs) of 8.36% is reported.

In the first set of numerical experiments, we choose the control group as $l_w = l_s = 0.3$, $h_w = 0.1$, $h_s = 0.5$, $b = 1$, $\lambda_w = \lambda_s = 2$, $T_w = 2$ and $T_s = 1$. Then we obtain nine treatment groups where each of them changes one parameter in the control group. Inventory costs savings are computed for the control group and all treatment groups. Comparisons between the two groups are depicted in Figure 2.

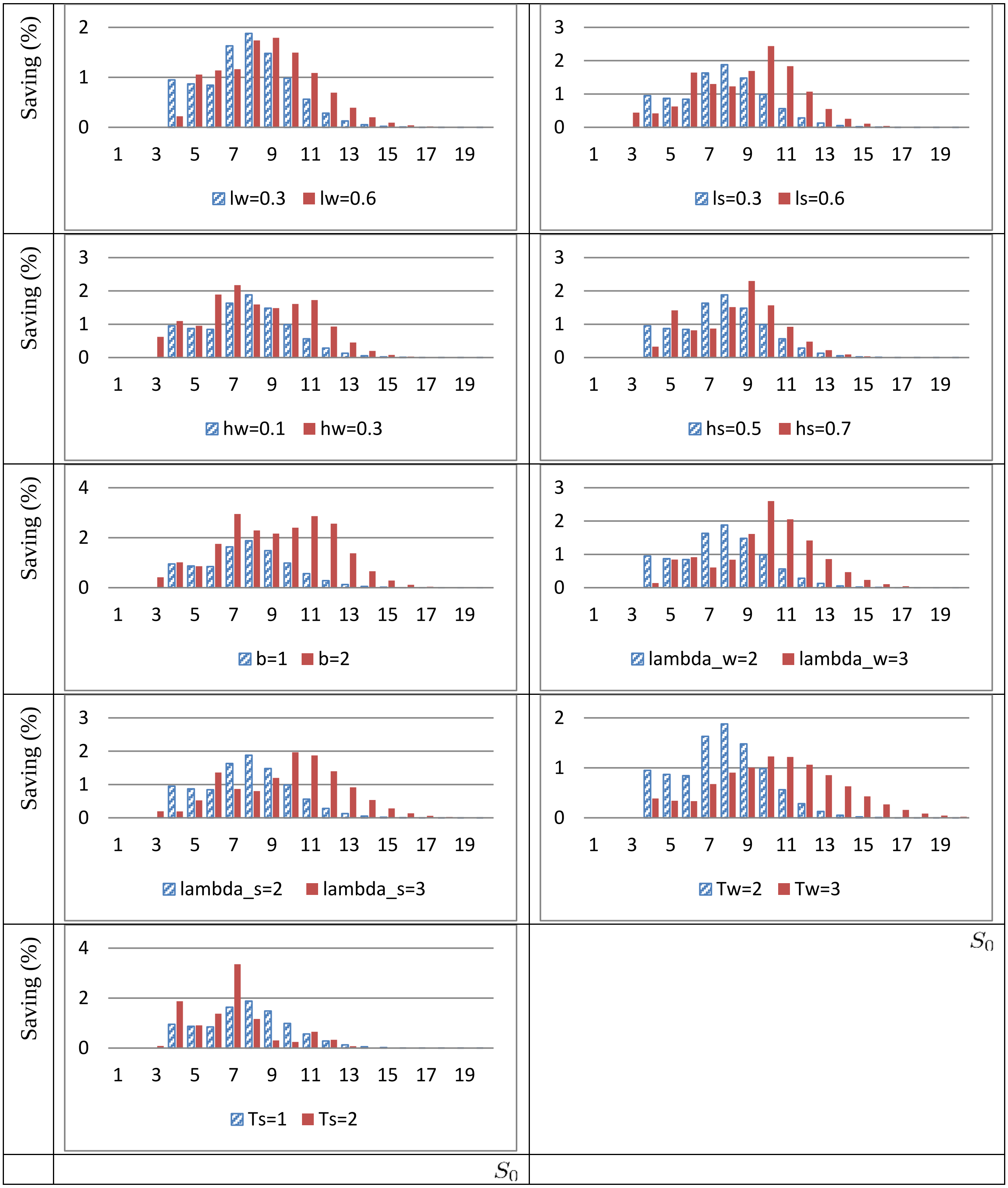


*Figure 2 – Comparisons between the control and the treatment group*

We have two observations from Figure 2. Firstly, there is no evident pattern between the two groups, i.e., for the same value of $S_0$, a larger parameter does not necessarily lead to more costs saving. Secondly, an inventory costs saving is bell shaped in $S_0$ and the dynamic policy leads to saving when $S_0$ is moderate and this is explained as follows. A moderate $S_0$ renders a static policy to allocate so much stock to the store that the warehouse incurs high backordering cost, whereas a dynamic policy balances the stock at the warehouse and the store at each review epoch.

In the second set, we vary $b$ from 1 to 10 and keep all other parameters as in the control group of the first set. For each value of $b$, we compute the maximum inventory costs saving among $S_0$. We find the maximum saving to be increasing in $b$, as is illustrated in Figure 3. It implies that as $b$ increases, inventory costs saving can be even larger than the reported maximum, which makes our dynamic policy even more attractive.

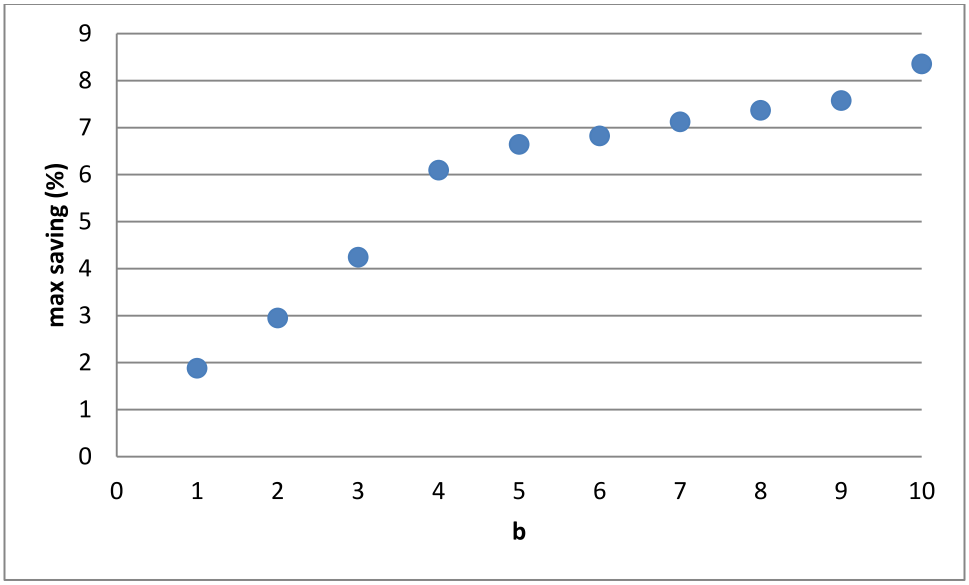


*Figure 3 – Maximum inventory costs saving increases in $b$*

## Summary


In this paper, we study stock allocation in a two-echelon dual-channel system which implements periodic reviews. We find that if online and walk-in demand backordering cost rates are equal, then store order-up-levels at different review epochs can be decided independently. A dynamic policy is proposed and the optimal one is guaranteed to outperform any static policy. The convexity of the cost function makes it easy to compute the optimal order-up-to levels at each review epoch. Our numerical results indicate that our dynamic policy lead to an 8.36% saving in system inventory costs and the dynamic policy is most appropriate when system inventory level is moderate and the backordering cost rate is high.


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