Valuing Prearranged Paired Kidney Exchanges: A Stochastic Game Approach

Murat Kurt {muratkur@buffalo.edu}

Department of Industrial and Systems Engineering

University at Buffalo, The State University of New York

415 Bell Hall, University at Buffalo North Campus, Buffalo, NY-14260

Andrew J. Schaefer {schaefer@ie.pitt.edu}

Department of Industrial Engineering

University of Pittsburgh

3700 O'Hara Street, 1048 Benedum Hall, Pittsburgh, PA-15261

M. Utku Ünver {unver@bc.edu}

Department of Economics

Boston College

140 Commonwealth Avenue, 21 Campanella Way, Boston, MA-02467

Mark S. Roberts {mroberts@pitt.edu}

Department of Health Policy and Management

University of Pittsburgh

130 De Soto Street, A620 Crabtree Hall, Pittsburgh, PA-15261

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Abstract

End-stage renal disease (ESRD) is the ninth-leading cause of death in the U.S. Transplantation is the most viable renal replacement therapy for ESRD patients, but there is a severe disparity between the demand for kidneys for transplantation and the supply. This shortage is further complicated by incompatibilities in blood-type and antigen matching between patient-donor pairs. Paired kidney exchange (PKE), a cross-exchange of kidneys among incompatible patient-donor pairs, overcomes many difficulties in matching patients with incompatible donors. In a typical PKE, transplantation surgeries take place simultaneously so that no donor may renege after her intended recipient receives the kidney. Therefore, in a PKE, the occurrence of a transplantation requires compatibility among the pairs' willingnesses to exchange. We consider an arbitrary number of autonomous patients with probabilistically evolving health statuses in a prearranged PKE, and model their transplant timing decisions as a discrete-time non-zero-sum noncooperative stochastic game. We explore necessary and sufficient conditions for patients' decisions to be a stationary-perfect equilibrium, and formulate a mixed-integer linear programming representation of equilibrium constraints, which provides a characterization of the socially optimal stationary-perfect equilibria. We carefully calibrate our model using large scale nationally representative clinical data, and empirically confirm that randomized strategies, which are less consistent with clinical practice and rationality of the patients, do not yield a significant social welfare gain over pure strategies. We also quantify the social welfare loss due to patient autonomy and demonstrate that maximizing the number of transplants may be undesirable. Our results highlight the importance of the timing of an exchange and the disease severity on matching patient-donor pairs.

Keywords: medical decision making, paired kidney exchange, game theory, Markov decision processes, integer programming

1. Introduction

End-stage renal disease (ESRD) typically occurs when the kidneys' functionality is less than 10 % of normal [28]. More than 500,000 Americans have ESRD and the size of this population is expected to grow to 2.24 million by 2030 [54, 56].

There are two viable treatment alternatives for ESRD patients: Dialysis and transplantation. Due to insufficient supply of kidneys for transplantation dialysis is a common intermediary step. However, transplantation is the preferred choice of treatment as it provides improved long-term survival rates and a higher quality of life than dialysis [23, 60, 61]. In the U.S., an ESRD patient must join a waiting list administered by the United Network for Organ Sharing (UNOS) to be eligible for a cadaveric kidney transplantation. Because people can function normally on only one kidney, it is also possible for an ESRD patient to receive a kidney from a living-donor, and transplants from such donors generally yield better survival outcomes than those from cadaveric donors [17, 55]. Over 93,000 patients in the U.S. are currently awaiting a kidney transplant, but in 2010, only 15,430 patients received transplants, 5,700 of which were from living-donors [58].

Blood type and human leukocyte antigen (HLA) incompatibilities are the two main factors that make kidneys difficult to match, and due to such difficulties new clinical strategies have been proposed to alleviate the shortage of kidneys [26, 30, 42, 63]. A *paired kidney exchange* (PKE) is a cross-exchange of kidneys among incompatible patient-donor pairs (see Figure 1 for an illustration). PKEs typically reduce the waiting time for transplantation as well as the length of dialytic therapy, thereby reducing healthcare costs and productivity losses [24, 25, 43, 44]. PKEs have grown rapidly over the last two decades to overcome the difficulties in matching kidneys [46], and it has been estimated that they can raise the number of transplants up to 90 % [38]. The significant potential of PKEs [36, 46, 47] has also led to the establishment of several regional kidney exchange clearinghouses in the U.S. that expand the pool of living-donors by organizing the registry of patients and donors [5, 29, 57].

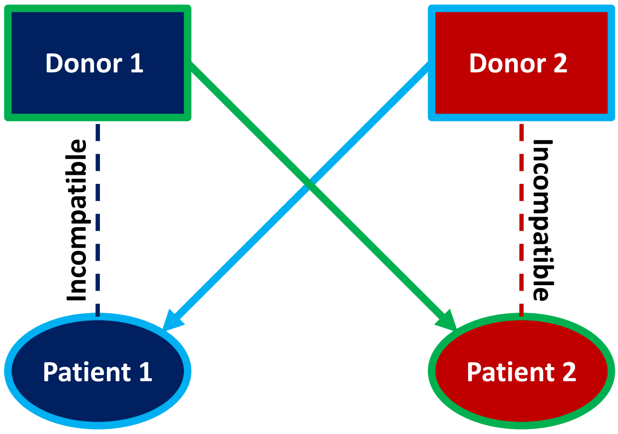

Figure 1: An illustration of a two-way kidney exchange. The donors are incompatible with their intended recipients, but Donor 1 is compatible with Patient 2 and Donor 2 is compatible with Patient 1.

PKEs have been on the focus of several studies, and mainly analyzed as a matching and/or a mechanism design problem [1, 35, 37, 59]. There have also been various efforts in the medical literature to weigh the relative merits and shortcomings of current PKE practice [11, 15, 16, 27, 36, 39, 40, 43, 45]. The Operations Research literature on modeling organ allocation decisions can be classified in three main strands. Research from an individual patient's perspective focuses on how an individual patient should act within a given organ allocation scheme [3, 4, 8, 21, 22, 41]. Research from the social perspective seeks organ allocation schemes to maximize one or more social objectives [9, 10, 33, 64, 65, 66, 67]. Lastly, the joint perspective recognizes possibly conflicting interests of the society and individual patients, but there has not been any discussion of the timing of transplantation among the pairs who have been already matched [51, 52, 53]. Also, in almost all PKEs, all exchange surgeries take place simultaneously, so that no donor may renege after her intended recipient receives the kidney [32, 34]. For this reason, in a PKE, the occurrence of a transplantation requires compatibility among the pairs' willingnesses to exchange and existing optimal organ transplantation timing models in the literature do not apply in this context.

While potential kidney exchanges have been formulated as finding matchings in a graph [37] there has not been any emphasis of the timing of transplantation and the effects of disease severity on patients' decisions. In this paper, we consider an arbitrarily sized group of matched incompatible patient-donor pairs for whom the only feasible exchange of kidneys is a cyclic exchange. Because the patients' health statuses are dynamic, and transplantation surgeries take place simultaneously, we model the patients' transplant timing decisions as a noncooperative stochastic game. In Section 2, we present the stochastic game formulation for which we present our equilibrium analyses in Section 3. In Section 4, we gain insights by illustrating our model with real clinical data for a large, nationally representative cohort. Finally, in Section 5 we conclude the paper by highlights.

2. Model Formulation

We consider $N \geq 2$ self-interested patient-donor pairs where Patient i is compatible for an exchange with Donor $i+1$ for $i = 1, 2, ..., N-1$ and Patient N is compatible with Donor 1. Periodically, each patient must decide whether to offer to exchange or wait, as her health evolves stochastically, and an exchange occurs only if all patients offer to exchange. After making the decision, each patient receives a reward that depends on her current health status. If an exchange occurs, each patient receives a lump-sum terminal reward (e.g. quality-adjusted post-transplant survival) and terminates the process; otherwise, she accrues an intermediate reward and revisits the same decision subsequently. We assume the kidney quality of each donor is static over time and that once a patient dies, her donor will not donate her kidney, rendering an exchange for the other patients infeasible. Furthermore, if a patient dies prior to an exchange, the other patients will never receive a transplant.

We consider an infinite decision horizon with discrete, equidistant time periods (e.g. daily or weekly) and model the transplant timing decisions that patients face in such a prearranged PKE as a non-zero-sum stochastic game, $\mathcal{G}$. We represent the set of patients in the exchange cycle by $\mathcal{N} = \{1, 2, ..., N\}$, and for any Patient $i \in \mathcal{N}$, we let $\mathcal{N}_{-i} = \mathcal{N} \setminus \{i\}$. Also, when $\mathcal{N} = \{1, 2\}$, for $i \in \mathcal{N}$, we let subscript $-i$ refer to j, where $j \in \mathcal{N} \setminus \{i\}$. We describe the components of game $\mathcal{G}$, all of which are assumed to be completely observable by each patient, as follows:

States: The state of the system is an ordered N-tuple of the patients' individual health states, $\mathbf{s} = (s_1, s_2, ..., s_N) \in \mathscr{S}$, where for each $i \in \mathcal{N}$, $s_i \in \Omega$ denotes the health state of Patient i, $\Omega = \{1, ..., S\}$ (with $S < \infty$) refers to the set of health states for each individual patient, and $\mathscr{S} = \Omega^N$

is the system's state space. For any patient $i \in \mathcal{N}$, $s_i = S$ refers to the (absorbing) death state and $\Phi = \Omega \setminus \{S\}$ represents the set of living states. We denote the set of states in which at least one of the patients is dead by $\mathscr{D} = \mathscr{S} \setminus \Phi^N$.

Strategies: Non-zero-sum stochastic games typically admit a large number of equilibria in non-stationary strategies, which may be hard to implement in practice due to their time-nonhomogeneous structure. Stationary strategies can restrict patients' dynamic interactions by constraining each of them to choose her actions in a time-independent manner, and are more consistent with clinical practice.

When Patient i follows strategy $\mathbf{a}_i = \left[a_i(\mathbf{s})\right]_{\mathbf{s}\in\mathscr{S}}$, $a_i(\mathbf{s}) \in [0,1]$ refers to the probability that she offers to exchange whenever the game occupies state $\mathbf{s} \in \mathscr{S}$. For any Patient $i \in \mathcal{N}$, given the combination of the other patients' strategies $\mathbf{a}_{-i} = \left\{\mathbf{a}_j\right\}_{j\in\mathcal{N}_{-i}}$, we let $\mathbf{A} = (\mathbf{a}_i, \mathbf{a}_{-i})$ represent the resulting strategy profile. Also, for a strategy profile $(\mathbf{a}_i, \mathbf{a}_{-i})$ we let $(\mathbf{a}'_i, \mathbf{a}_{-i})$ denote the strategy profile that results from replacing $\mathbf{a}_i$ by a strategy $\mathbf{a}'_i$.

Rewards: We define $u_i(\mathbf{s}, 0)$ to be the *immediate reward* of Patient i (e.g quality-adjusted life days or weeks) accrued in state $\mathbf{s} \in \mathscr{S}$ given an exchange does not occur. We also define $u_i(\mathbf{s}, 1)$ as the *expected post-transplant reward* (e.g. expected quality-adjusted post-transplant survival) of Patient $i \in \mathcal{N}$ given an exchange occurs in $\mathbf{s} \in \mathscr{S}$. Note that for each patient $i \in \mathcal{N}$ and state $\mathbf{s} \in \mathscr{S}$, $u_i(\mathbf{s}, 1)$ is a one-time lump-sum reward, where for each $i \in \mathcal{N}$, $u_i(\mathbf{s}, 1) = 0$ for all $s \in \mathscr{D}$ as there is no possibility for an exchange in such states for the surviving patients. We assume that each patient $i \in \mathcal{N}$ discounts her future rewards by a factor $\lambda_i \in (0, 1)$.

Probabilities: The state of the system evolves stochastically until an exchange occurs, or at least one of the patients dies, whichever occurs sooner, where each patient's health evolves according to a discrete-time finite-state Markov chain independent of the others'. Given an exchange does not occur in state $\mathbf{s} = (s_1, s_2, ..., s_N) \in \mathscr{S}$, the system moves to state $\mathbf{s}' = (s'_1, s'_2, ..., s'_N) \in \mathscr{S}$ with probability $\mathscr{P}(\mathbf{s}'|\mathbf{s}) = \prod_{i\in\mathcal{N}} P_i(s'_i|s_i)$, where $P_i(s'_i|s_i)$ denotes the probability that Patient $i \in \mathcal{N}$ will be in health state $s'_i \in \Omega$ at epoch $t+1$ given she is in state $s_i \in \Omega$ at epoch t.

In the remainder of the paper, we let the terms in bold refer to a real-valued hypermatrix, *i.e.*, $\mathbf{v}$ refers to the hypermatrix $\big[v(\mathbf{s})\big]_{\mathbf{s}\in\mathscr{S}}$. For convenience, for each patient $i \in \mathcal{N}$, given a real-valued hypermatrix $\mathbf{v}$ and state $\mathbf{s} \in \mathscr{S}$, we let $F_i(\mathbf{s}, \mathbf{v}) = u_i(\mathbf{s}, 0) + \lambda_i \sum_{\mathbf{s}'\in\mathscr{S}} \mathscr{P}(\mathbf{s}'|\mathbf{s})v(\mathbf{s}')$. We interpret $F_i(\mathbf{s}, \mathbf{v})$ as the total expected discounted reward of Patient i given an exchange does not occur starting in state $\mathbf{s} \in \mathscr{S}$ and the underlying strategy profile induces $\mathbf{v}$ as her expected rewards.

We let $g_i(\mathbf{s}, \mathbf{a}_i, \mathbf{a}_{-i})$ represent the *total expected discounted payoff* (e.g. total discounted quality-adjusted life expectancy) for Patient $i \in \mathcal{N}$ starting in state $\mathbf{s} \in \mathscr{S}$ under strategy profile $\mathbf{A}$. Note that under strategy profile $\mathbf{A}$, an exchange occurs in state $\mathbf{s} \in \mathscr{S}$ with probability $\prod_{j\in\mathcal{N}} a_j(\mathbf{s})$. Then, because $F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})\big)$ represents the expected future reward of Patient i given an exchange does not occur immediately when the system starts in state $\mathbf{s}$ under strategy profile $(\mathbf{a}_i, \mathbf{a}_{-i})$, the payoffs are recursively defined as follows:

$$g_i(\mathbf{s}, \mathbf{a}_i, \mathbf{a}_{-i}) = u_i(\mathbf{s}, 1) \prod_{j\in\mathcal{N}} a_j(\mathbf{s}) + \left(1 - \prod_{j\in\mathcal{N}} a_j(\mathbf{s})\right) F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})\big) \text{ for } \mathbf{s} \in \mathscr{S},\ i \in \mathcal{N}. \tag{1}$$

We assume that game $\mathcal{G}$ is of perfect recall so that each patient has a perfect memory of her previous actions and those of all other patients. Furthermore, each patient is assumed to behave rationally only for her self-interest during the course of the game. In the rest of the paper, we represent componentwise relations between given two hypermatrices in matrix notation. For instance, given $\mathbf{v}_1$ and $\mathbf{v}_2$, $\mathbf{v}_1 = \mathbf{v}_2$ refers to $v_1(\mathbf{s}) = v_2(\mathbf{s})$ for all $\mathbf{s} \in \mathscr{S}$.

3. Equilibrium Analysis

In this section, we provide an extensive equilibrium analysis of game $\mathcal{G}$. *Nash equilibrium* is the most commonly used solution concept to analyze the outcomes of noncooperative games [14]. In our context, in a Nash equilibrium, no patient may gain by changing her own strategy unilaterally. Because our focus is on stationary strategies, our analyses involve only Nash equilibria that are in stationary strategies, known as *stationary equilibria*. A strategy profile $\mathbf{A}$ is a stationary equilibrium of game $\mathcal{G}$ if for all $i \in \mathcal{N}$:

$$\mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i}) \geq \mathbf{g}_i(\mathbf{a}'_i, \mathbf{a}_{-i}) \text{ for all } \mathbf{a}'_i. \tag{2}$$

Note that a strategy profile satisfying (2) is a Nash equilibrium of game $\mathcal{G}$ independent from the initial state of the game, *i.e.*, neither of the patients can be better off in any state $\mathbf{s} \in \mathscr{S}$ by only changing her strategy unilaterally. Therefore, strategy profiles satisfying (2) are also known as *stationary-perfect equilibria* in the economics literature. In the remainder of the paper, unless otherwise stated, we let the terms "strategy" and "equilibrium" refer to "stationary strategy" and "stationary-perfect equilibrium", respectively.

Recall that an exchange can occur in state $\mathbf{s} \in \mathscr{S}$ only if all patients choose a positive probability to offer to exchange. Therefore, a single patient can not affect the outcome as long as one of the other patients chooses to wait. As an intuitive consequence, Theorem 1 provides necessary and sufficient conditions for a strategy profile to be an equilibrium of game $\mathcal{G}$.

Theorem 1 : *A strategy profile* $\mathbf{A}$ *is an equilibrium of game* $\mathcal{G}$ *if and only if for all* $\mathbf{s} \in \mathscr{S}$ *and* $i \in \mathcal{N}$*:*

$$\begin{aligned} g_i(\mathbf{s}, \mathbf{a}_i, \mathbf{a}_{-i}) = \max \Bigg\{ & F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})\big), \\ & u_i(\mathbf{s}, 1) \prod_{j \in \mathcal{N}_{-i}} a_j(\mathbf{s}) + \Bigg(1 - \prod_{j \in \mathcal{N}_{-i}} a_j(\mathbf{s})\Bigg) F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})\big) \Bigg\}. \end{aligned} \tag{3}$$

Because Nash equilibria are immune only to unilateral deviations, game $\mathcal{G}$ may admit a large number of pathological equilibria that make little clinical sense. For instance, by Theorem 1, any strategy profile under which at least two arbitrary, but not necessarily the same, patients offer to exchange with probability 0 in every state $\mathbf{s} \in \mathscr{S}$ denotes an equilibrium of game $\mathcal{G}$. Furthermore, as different equilibria may imply different payoff outcomes, due to vast multiplicity of equilibria that game $\mathcal{G}$ can admit, a complete characterization of such equilibria is computationally prohibitive. As such, we consider equilibrium selection and motivate the following question: *Given the game starts in state* $\widehat{\mathbf{s}} \in \mathscr{S}$*, which equilibrium maximizes the social welfare, i.e., the sum of the patients' total expected payoffs?* Specifically, we let Γ denote the set of equilibria of game $\mathcal{G}$, and are interested in the equilibria of game $\mathcal{G}$ that represents an optimal solution to: $\max_{\mathbf{A}\in\Gamma}\left[\sum_{i\in\mathcal{N}} g_i(\widehat{\mathbf{s}}, \mathbf{a}_i, \mathbf{a}_{-i})\right]$. In the remainder of the paper, we will refer to a social welfare maximizing equilibrium a "socially optimal equilibrium".

Equilibria of non-zero-sum discounted stochastic games with finite state and/or action spaces can be characterized by mathematical programs. The problem of computing an equilibrium of a finite discounted stochastic game is equivalent to finding the global optima of a nonlinear program with linear constraints [7, 12, 13]. The computation of the global optima of such mathematical programs requires the construction of algorithms that are free of convergence problems [6, 19, 20, 31]. However, for general non-zero-sum discounted stochastic games there is no known way of selecting a *best* equilibrium with respect to a given optimality criteria, short of enumeration. Therefore, from an algorithmic point of view our equilibrium selection requires a deeper exploration of the underlying model structure.

Lemma 1 (i) states that in an equilibrium, an exchange can not occur in a particular state $\mathbf{s} \in \mathscr{S}$ as long as there are some patients who are strictly better off waiting in that state. Therefore, by Lemma 1 (ii), in an equilibrium, a patient strictly randomizes between waiting and offering to exchange in state $\mathbf{s} \in \mathscr{S}$, only when she is indifferent between her expected post-transplant reward and expected payoff-to-go, and each of the other patients offers to exchange with some positive probability in that state. Note that by (1), statewise exchange occurrence probabilities sufficiently describe the payoff profile of a strategy profile, *i.e.*, two different strategy profiles $\mathbf{A}$ and $\mathbf{A}'$ induce the same payoff profile if $\prod_{j\in\mathcal{N}} a_j(\mathbf{s}) = \prod_{j\in\mathcal{N}} a'_j(\mathbf{s})$ for all $\mathbf{s} \in \mathscr{S}$. Therefore, for any equilibrium we can construct an alternative equilibrium with an equivalent payoff profile in which there is no more than a single patient who randomizes in any state $\mathbf{s} \in \mathscr{S}$, which we formally state in Lemma 1 (iii). For a given strategy profile $\mathbf{A}$, for $\mathbf{s} \in \mathscr{S}$, let $\mathscr{Y}_{\mathbf{s}}(\mathbf{A}) = \left\{i \in \mathcal{N} | a_i(\mathbf{s}) \in (0,1)\right\}$ denote the set of patients that randomize between waiting and offering to exchange in state $\mathbf{s}$.

Lemma 1 : *Suppose* $\mathbf{A} \in \Gamma$. *Then:*

(i) *For any* $\mathbf{s} \in \mathscr{S}$, *if* $\max\limits_{i\in\mathcal{N}}\left[g_i(\mathbf{s}, \mathbf{a}_i, \mathbf{a}_{-i}) - u_i(\mathbf{s}, 1)\right] > 0$ *then* $\prod_{j\in\mathcal{N}} a_j(\mathbf{s}) = 0$.

(ii) For any $\mathbf{s} \in \mathscr{S}$*, if* $\prod_{j\in\mathcal{N}} a_j(\mathbf{s}) \in (0,1)$ *then for any* $i \in \mathcal{N}$*,* $a_i(\mathbf{s}) \in (0,1)$ *implies* $u_i(\mathbf{s},1) = F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})\big)$*.*

(iii) There exists $\mathbf{A}' \in \Gamma$ *with* $\mathbf{g}_i(\mathbf{a}'_i, \mathbf{a}'_{-i}) = \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})$ *for all* $i \in \mathcal{N}$ *and* $|\mathscr{Y}_{\mathbf{s}}(\mathbf{A}')| \leq 1$ *for all* $\mathbf{s} \in \mathscr{S}$*.*

Next, we will present a family of mixed-integer linear programming (MIP) models to choose among equilibria. When there are two patients, these MIP models can find the socially optimal randomized or pure equilibria. When there are three or more patients, these MIP models can find the socially optimal pure equilibrium and provide an upper bound on the socially optimal randomized equilibrium. As an aside, our MIP models can also optimize over other objectives that are linear in the patients' expected payoffs. Our models require the set of hypermatrices $\{\mathbf{V}_i\}_{i\in\mathcal{N}}$, where for each patient $i \in \mathcal{N}$, $V_i(\mathbf{s}) = \max\{u_i(\mathbf{s},1), F_i(\mathbf{s},\mathbf{V}_i)\}$ for $\mathbf{s} \in \mathscr{S}$. The payoff matrix $\mathbf{V}_i$ represents the optimal value function of Patient i when the autonomy of the other patients are suppressed over the course of the game.

Lemma 2 : *For any Patient* $i \in \mathcal{N}$*,* $\mathbf{V}_i \geq \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})$ *for all* $\mathbf{A} \in \Gamma$*.*

For the decision variables $\mathbf{w} = \{\mathbf{w}_i\}_{i\in\mathcal{N}}$, $\mathbf{y} = \{\mathbf{y}_i\}_{i\in\mathcal{N}}$ and $\mathbf{z}$, we consider the following set of inequalities:

$$w_i(\mathbf{s}) \geq F_i(\mathbf{s}, \mathbf{w}_i) \qquad \forall \mathbf{s} \in \mathscr{S}, i \in \mathcal{N}, \tag{4a}$$

$$w_i(\mathbf{s}) \leq F_i(\mathbf{s}, \mathbf{w}_i) + u_i(\mathbf{s},1) y_i(\mathbf{s}) \qquad \forall \mathbf{s} \in \mathscr{S}, i \in \mathcal{N}, \tag{4b}$$

$$w_i(\mathbf{s}) \geq u_i(\mathbf{s},1)\big[y_j(\mathbf{s}) - y_i(\mathbf{s})\big] \qquad \forall \mathbf{s} \in \mathscr{S}, i \in \mathcal{N}, j \in \mathcal{N}_{-i}, \tag{4c}$$

$$w_i(\mathbf{s}) \geq \left(1 - N + \sum_{j\in\mathcal{N}} y_j(\mathbf{s})\right) u_i(\mathbf{s},1) \qquad \forall \mathbf{s} \in \mathscr{S}, i \in \mathcal{N}, \tag{4d}$$

$$w_i(\mathbf{s}) \leq u_i(\mathbf{s},1) z(\mathbf{s}) + V_i(\mathbf{s})\big[1 - z(\mathbf{s})\big] \qquad \forall \mathbf{s} \in \mathscr{S}, i \in \mathcal{N}, \tag{4e}$$

$$z(\mathbf{s}) \geq y_i(\mathbf{s}) \qquad \forall \mathbf{s} \in \mathscr{S}, i \in \mathcal{N}, \tag{4f}$$

$$(N-1) z(\mathbf{s}) \leq \sum_{i\in\mathcal{N}} y_i(\mathbf{s}) \qquad \forall \mathbf{s} \in \mathscr{S}, \tag{4g}$$

$$y_i(\mathbf{s}) \in \{0,1\} \qquad \forall \mathbf{s} \in \mathscr{S}, i \in \mathcal{N}, \tag{4h}$$

$$z(\mathbf{s}) \leq 1 \qquad \forall \mathbf{s} \in \mathscr{S}, \tag{4i}$$

$$w_i(\mathbf{s}) \geq 0 \qquad \forall \mathbf{s} \in \mathscr{S}, i \in \mathcal{N}. \tag{4j}$$

We denote $\Lambda := \{(\mathbf{w}, \mathbf{y}, \mathbf{z}) | (4a) - (4j)\}$. Given a feasible solution $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ to Λ, we interpret the values of the variables as follows: The variable $\widehat{\mathbf{w}}$ denotes a payoff profile, so that $F_i(\mathbf{s}, \widehat{\mathbf{w}}_i)$ denotes the expected payoff-to-go for Patient $i \in \mathcal{N}$ starting in state $\mathbf{s} \in \mathscr{S}$. The variable $\widehat{\mathbf{y}}$ denotes statewise randomization indicators, that is, in a particular state $\mathbf{s} \in \mathscr{S}$, $\widehat{y}_i(\mathbf{s}) = \ell$ for all $i \in \mathcal{N}$ means that the exchange occurrence probability is equal to ℓ in that state. Otherwise, *i.e.*, if $\widehat{y}_i(\mathbf{s}) \neq \widehat{y}_j(\mathbf{s})$ for some $i, j \in \mathcal{N}$, then any Patient $i \in \mathcal{N}$ with $\widehat{y}_i(\mathbf{s}) = 1$ offers to exchange with probability 1 whereas any Patient $j \in \mathcal{N}$ with $\widehat{y}_j(\mathbf{s}) = 0$ randomizes between waiting and offering to exchange. The variable $\widehat{\mathbf{z}}$ enforces logical relationships among the $\widehat{\mathbf{y}}_i$ variables for $i \in \mathcal{N}$.

Theorem 2 reveals the relationship between equilibria of game $\mathcal{G}$ and feasible solutions to Λ. For any equilibrium $\widehat{\mathbf{A}}$ of game $\mathcal{G}$, one can construct a feasible solution $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ to Λ from $\widehat{\mathbf{A}}$ in which $\widehat{\mathbf{w}}$ represents the payoff profile of $\widehat{\mathbf{A}}$. As a special case, when there are two patient-donor pairs, for any pair $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ satisfying (4a)-(4j) one can construct an equilibrium $\widehat{\mathbf{A}}$ of game $\mathcal{G}$ from $\widehat{\mathbf{w}}$ and $\widehat{\mathbf{y}}$ with a payoff profile equivalent to $\widehat{\mathbf{w}}$. Note that for a feasible solution $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ to Λ, while $\widehat{\mathbf{w}}$ represents the payoff profile of the equilibrium that $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ induces, as $\widehat{\mathbf{y}}$ involve only binary values, it may not necessarily represent the strategies of the resulting equilibrium.

Theorem 2 : *(i) For any $\widehat{\mathbf{A}} \in \Gamma$, there exists $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) \in \Lambda$ with $\widehat{\mathbf{w}}_i = \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})$ for all $i \in \mathcal{N}$.*

(ii) For any $\widehat{\mathbf{s}} \in \mathscr{S}$, $\max\limits_{\mathbf{A} \in \Gamma} \left(\sum_{i \in \mathcal{N}} g_i(\widehat{\mathbf{s}}, \mathbf{a}_i, \mathbf{a}_{-i}) \right) \leq \max\limits_{(\mathbf{w}, \mathbf{y}, \mathbf{z}) \in \Lambda} \left(\sum_{i \in \mathcal{N}} w_i(\widehat{\mathbf{s}}) \right)$.

(iii) If $N = 2$, for any $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) \in \Lambda$, there exists $\widehat{\mathbf{A}} \in \Gamma$ with $\widehat{\mathbf{w}}_i = \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})$ for both $i \in \mathcal{N}$.

(iv) If $N = 2$, then for any $\widehat{\mathbf{s}} \in \mathscr{S}$, $\max\limits_{\mathbf{A} \in \Gamma} \left(\sum_{i \in \mathcal{N}} g_i(\widehat{\mathbf{s}}, \mathbf{a}_i, \mathbf{a}_{-i}) \right) = \max\limits_{(\mathbf{w}, \mathbf{y}, \mathbf{z}) \in \Lambda} \left(\sum_{i \in \mathcal{N}} w_i(\widehat{\mathbf{s}}) \right)$.

As randomized strategies may involve the selection of statewise probability distributions over waiting and offering to exchange, pure strategies specify deterministic actions in each state and therefore have a more intuitive appeal to patients and physicians for clinical practice. Theorem 3 (*i*) derives necessary and sufficient conditions for a strategy profile to be a pure equilibrium of game $\mathcal{G}$. Theorem 3 (*ii*) draws on Theorem 2 (*i*) and Theorem 3 (*i*), and refines the set of solutions to the constraints (4a)-

(4j) to consider the issue of equilibrium selection within the class of pure equilibria. Then, Theorem 3 (*iii*) states that an optimal pure equilibrium with respect to a given criteria (which is linear in patients' expected payoffs) can be characterized as an optimal solution to this refinement. Lastly, Theorem 3 (*iv*) reveals the influence of patient autonomy on the payoffs of a socially optimal equilibrium. It states that in a socially optimal pure equilibrium, either an immediate exchange is optimal or at least one of the patients benefits from delaying the exchange. In the remainder of the paper, we let Π denote the set of pure equilibria of game $\mathcal{G}$, that is, $\Pi := \big\{\mathbf{A} \in \Gamma | a_i(\mathbf{s}) \in \{0,1\} \text{ for all } \mathbf{s} \in \mathscr{S} \text{ and } i \in \mathcal{N}\big\}$. We also let $\Upsilon := \big\{(\mathbf{w},\mathbf{y},\mathbf{z}) \in \Lambda | \mathbf{y}_{i+1} = \mathbf{y}_i \ \text{ for } i \in \mathcal{N}_{-N}\big\}$.

Theorem 3 : (*i*) *A strategy profile* $\mathbf{A}$ *is a pure equilibrium of game* $\mathcal{G}$ *if and only if for all* $\mathbf{s} \in \mathscr{S}$ *and* $i \in \mathcal{N}$,

$$a_i(\mathbf{s}) \in \begin{cases} \{1\} & \textit{if } \prod_{j\in\mathcal{N}_{-i}} a_j(\mathbf{s}) = 1 \textit{ and } u_i(\mathbf{s},1) \geq F_i\big(\mathbf{s},\mathbf{g}_i(\mathbf{a}_i,\mathbf{a}_{-i})\big) \\ \{0\} & \textit{if } \prod_{j\in\mathcal{N}_{-i}} a_j(\mathbf{s}) = 1 \textit{ and } u_i(\mathbf{s},1) < F_i\big(\mathbf{s},\mathbf{g}_i(\mathbf{a}_i,\mathbf{a}_{-i})\big) \\ \{0,1\} & \textit{otherwise.} \end{cases} \tag{5}$$

(*ii*) $(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}}) \in \Upsilon$ *if and only if there exists* $\widehat{\mathbf{A}} \in \Pi$ *with* $\widehat{\mathbf{w}}_i = \mathbf{g}_i(\widehat{\mathbf{a}}_i,\widehat{\mathbf{a}}_{-i})$ *for all* $i \in \mathcal{N}$.

(*iii*) *Given* $c_i(\mathbf{s}) \in \mathbb{R}$ *for all* $\mathbf{s} \in \mathscr{S}$ *and* $i \in \mathcal{N}$:

$$\max_{\mathbf{A}\in\Pi}\left(\sum_{\mathbf{s}\in\mathscr{S},i\in\mathcal{N}} c_i(\mathbf{s})g_i(\mathbf{s},\mathbf{a}_i,\mathbf{a}_{-i})\right) = \max_{(\mathbf{w},\mathbf{y},\mathbf{z})\in\Upsilon}\left(\sum_{\mathbf{s}\in\mathscr{S},i\in\mathcal{N}} c_i(\mathbf{s})w_i(\mathbf{s})\right).$$

(*iv*) *Given* $\widehat{\mathbf{s}} \in \mathscr{S}$, *let* $\mathbf{A}^* \in \arg\max_{\mathbf{A}\in\Pi}\big[\sum_{i\in\mathcal{N}} g_i(\widehat{\mathbf{s}},\mathbf{a}_i,\mathbf{a}_{-i})\big]$. *Then, either* $g_i(\widehat{\mathbf{s}},\mathbf{a}_i^*,\mathbf{a}_{-i}^*) = u_i(\widehat{\mathbf{s}},1)$ *for all* $i \in \mathcal{N}$ *or* $\max_{i\in\mathcal{N}}\big[g_i(\widehat{\mathbf{s}},\mathbf{a}_i^*,\mathbf{a}_{-i}^*) - u_i(\widehat{\mathbf{s}},1)\big] > 0$.

We can interpret (5) as follows. In a pure equilibrium of game $\mathcal{G}$, for Patient i there are two possible scenarios in each state $\mathbf{s} \in \mathscr{S}$: If some of the other patients want to wait, then because the decision of Patient i will not affect the occurrence of the exchange, she is indifferent between waiting and offering to exchange. Otherwise, she offers as well only if she benefits from the exchange.

4. Numerical Study

We illustrate our model using clinical data. As most of the social benefit is accrued by exchanges with three or fewer patients [38], we restrict our focus to two- and three-way exchanges. For convenience and consistency on notation, we present results only from two-way exchanges and describe results on three-way exchanges in the appendix. While maximizing the social objective, we estimate the cost of restricting our attention to pure equilibria, rather than randomized equilibria. After demonstrating that this cost appears to be negligible, we consider pure equilibria for the rest of the experiments.

4.1 Data Sources and Parameter Estimation

In this section, we estimate the transition probabilities and post-transplant rewards based on clinical data. There is a broad consensus among clinicians that glomerular filtration rate (GFR) is the best measure of remaining pre-dialysis kidney functionality for ESRD patients. Although the stages of ESRD are mainly based on measured or estimated GFR [28], it appears that no stochastic model of pre-dialysis GFR progression has been described in the literature. We use GFR levels and the patient's dialysis status to represent her health. To build a Markovian progression of pre-dialysis GFR levels, we use data set from The Thomas E. Starzl Transplantation Institute at the University of Pittsburgh Medical Center (UPMC), one of the largest transplantation centers nationwide. This set provides detailed data on laboratory measurements for more than 60,000 ESRD patients, but due to limited availability of data for some ethnicities, we focus our experiments to Caucasian and African-American patients. We discretize the continuous range of GFR levels into 10 ranges, the boundaries of which we present in Table 1.

We assume that the patient gets on dialysis whenever her GFR falls below 15 mL/min/1.73 m^2 [28] and that once the patient initiates dialysis she can not recover her renal functionality prior to receiving a transplant [2]. We add the absorbing death state to the set of GFR ranges so that $\Omega = \{1, 2, ..., 11\}$ refers to the set of health states for each individual patient. Because the UPMC

Table 1: Boundaries of GFR ranges (in mL/min/1.73 m^2) for numerical experiments.

	GFR Range									
Boundary	1	2	3	4	5	6	7	8	9	10
Lower Bound	60	50	40	30	25	22.5	20	17.5	15	0
Upper Bound	∞	60	50	40	30	25	22.5	20	17.5	15

data set has sparsely available GFR data, in an approach similar to Shechter [48], for each patient we use a shape-preserving piecewise Hermitian cubic spline to interpolate her missing laboratory measurements on a daily basis. We let $N_s(s')$ denote the number of patients whose GFR moves from range $s \in \Phi$ into $s' \in \Phi$ on any two successive days. Then, we define $q(s'|s)$ to be the patient's probability of having a GFR in range $s' \in \Phi$ on day $h+1$ given she had a GFR in range $s \in \Phi$ on day h. Because the patient does not recover her renal functionality after dialysis, $q(s'|S-1) = 0$ for all $s' \in \Phi \setminus \{S-1\}$ and $q(S-1|S-1) = 1$. We calculate the probability of moving from a pre-dialysis GFR range $s \in \Phi$ into GFR range $s' \in \Phi$ as $q(s'|s) = N_s(s')\left(\sum_{s' \in \Phi} N_s(s')\right)^{-1}$. Note that the UPMC data set does not have complete patient mortality information. Therefore, the probabilities $\{q(s'|s)\}_{\{s,s' \in \Phi\}}$ are conditional on the patient's survival. We denote the patient's death probability in GFR range $s \in \Phi$ by $\varphi(s)$. We use the U.S. Renal Data System (USRDS) [56] to estimate the patient's death probability on dialysis and GFR-based mortality rates from Go et al. [18] to estimate the patient's death probabilities in pre-dialysis stage (see Figure 2 for a summary of data sources and references). Then, we define the probabilities governing daily transitions among a single patient's health states as:

$$\widetilde{P}(s'|s) = \begin{cases} \varphi(s) & \text{for } s \in \Phi \text{ and } s' = S, \\ \big[1 - \varphi(s)\big] q(s'|s) & \text{for } s \in \Phi \text{ and } s' \in \Phi, \\ 1 & \text{for } s = s' = S, \\ 0 & \text{for } s = S \text{ and } s' \in \Phi. \end{cases}$$

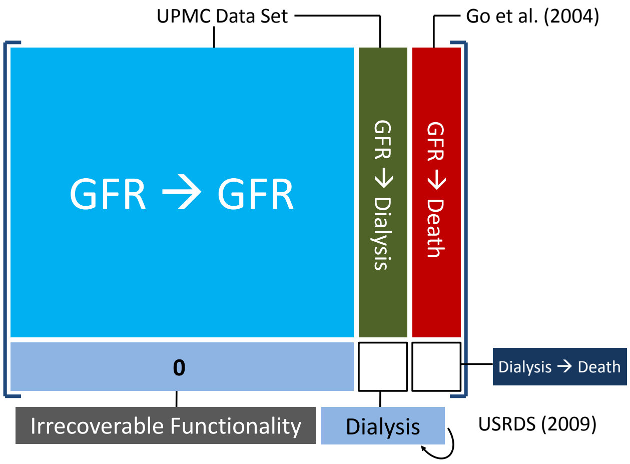

Figure 2: The data sources and references used in estimating daily-based health transition matrices

There are two types of rewards in our model: Immediate rewards, $u_i(\mathbf{s}, 0)$, and expected post-transplant rewards, $u_i(\mathbf{s}, 1)$. Although these rewards are defined as a function of the state of the game, because patients' health statuses are independent of each other, we define the rewards of the game mainly in terms of patients' individual health states. We define $r(s, 0)$ to be the immediate reward in days that patient accrues if she waits in state $s \in \Omega$, and $r(s, 1)$ to be the expected post-transplant reward in days when she receives a transplant in state $s \in \Omega$. Following recent literature in kidney transplantation [23, 62], to account for the adverse side effects of dialysis on the patient's quality of life we assume a quality-adjustment factor 0.8 for being on dialysis; that is, a year spent on dialysis is assumed to be equivalent to 0.8 years of life with full quality. Therefore, for the immediate rewards we define:

$$r(s, 0) = \begin{cases} 1 & \text{if } s \in \{1, ..., S-2\}, \\ 0.8 & \text{if } s = S-1, \\ 0 & \text{if } s = S. \end{cases}$$

We use available post-transplant survival rates from Scientific Registry of Transplant Recipients (SRTR) [50] to estimate expected post-transplant survivals. We develop a proportional hazards model and use the associated risk adjustment coefficients from SRTR to estimate the patient's quality-adjusted expected post-transplant rewards [49]. Due to limited availability of data, we assume exponential growth in death rate to extrapolate the survival rates which are available only for short-term. We set biweekly decision epochs and apply a 0.97 annual discount rate for each patient (daily $\lambda' \cong 0.999916$ and biweekly $\lambda \cong 0.998829$). Since our transition probability estimations are on daily basis and decisions are revisited every two weeks, we adjust the daily parameters to biweekly basis.

We denote the matrix that governs biweekly GFR transitions by $\mathbf{P} = \widetilde{\mathbf{P}}^{14}$. We also define $\psi^t(s)$ as the total expected discounted immediate reward in quality-adjusted life days that will be accrued in the last $14 - t$ days of the current stage, and recursively calculate it as:

$$\psi^t(s) = \begin{cases} r(s,0) + \lambda' \sum_{s' \in \Omega} \widetilde{P}(s'|s)\psi^{t+1}(s') & \text{for } s \in \Omega \text{ and } t = 0, ..., 13, \\ 0 & \text{for } s \in \Omega \text{ and } t = 14. \end{cases}$$

After we discount the expected post-transplant rewards and specify the rewards by a subscript for Patient i, we define the rewards of the game as follows:

$$u_i(\mathbf{s}, m) = \begin{cases} \psi_i^0(s_i) & \text{for } \mathbf{s} = (s_1, ..., s_i, ..., s_N) \in \mathscr{S} \text{ and } m = 0, \\ r_i(s_i, 1) & \text{for } \mathbf{s} = (s_1, ..., s_i, ..., s_N) \in \mathscr{S} \setminus \mathscr{D} \text{ and } m = 1, \\ 0 & \text{for } \mathbf{s} = (s_1, ..., s_i, ..., s_N) \in \mathscr{D} \text{ and } m = 1. \end{cases}$$

For our experiments, we simulate 2500 two-way and 500 three-way exchange cases by using the average frequency of patient-donor characteristics in living-donor kidney transplantations that are available from SRTR for the years 1998-2007. We simulate the patients' initial GFR ranges uniformly over Φ.

4.2 Welfare Loss Due to Patient Autonomy and Equilibrium Selection

We solve our MIP models to characterize socially optimal randomized and pure equilibria. For each exchange case we solve the mathematical programs

$$\max_{(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}}) \in \Lambda} \left(\sum_{i \in \mathcal{N}} w_i(\widehat{\mathbf{s}}) \right) \quad \text{and} \quad \max_{(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}}) \in \Upsilon} \left(\sum_{i \in \mathcal{N}} w_i(\widehat{\mathbf{s}}) \right)$$

to estimate the cost of restricting our attention to pure strategies. By Theorems 2 and 3,

$$\max_{(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}}) \in \Lambda} \left(\sum_{i \in \mathcal{N}} w_i(\widehat{\mathbf{s}}) \right) - \max_{(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}}) \in \Upsilon} \left(\sum_{i \in \mathcal{N}} w_i(\widehat{\mathbf{s}}) \right)$$

quantifies the aforementioned cost exactly for two-way kidney exchanges. Our experiments reveal that a socially optimal paired pure equilibrium is only negligibly worse than a socially optimal randomized equilibrium. In all our instances, the socially optimal pure equilibrium was never more than 0.01% worse than the socially optimal randomized equilibrium. Therefore, as randomized strategies are

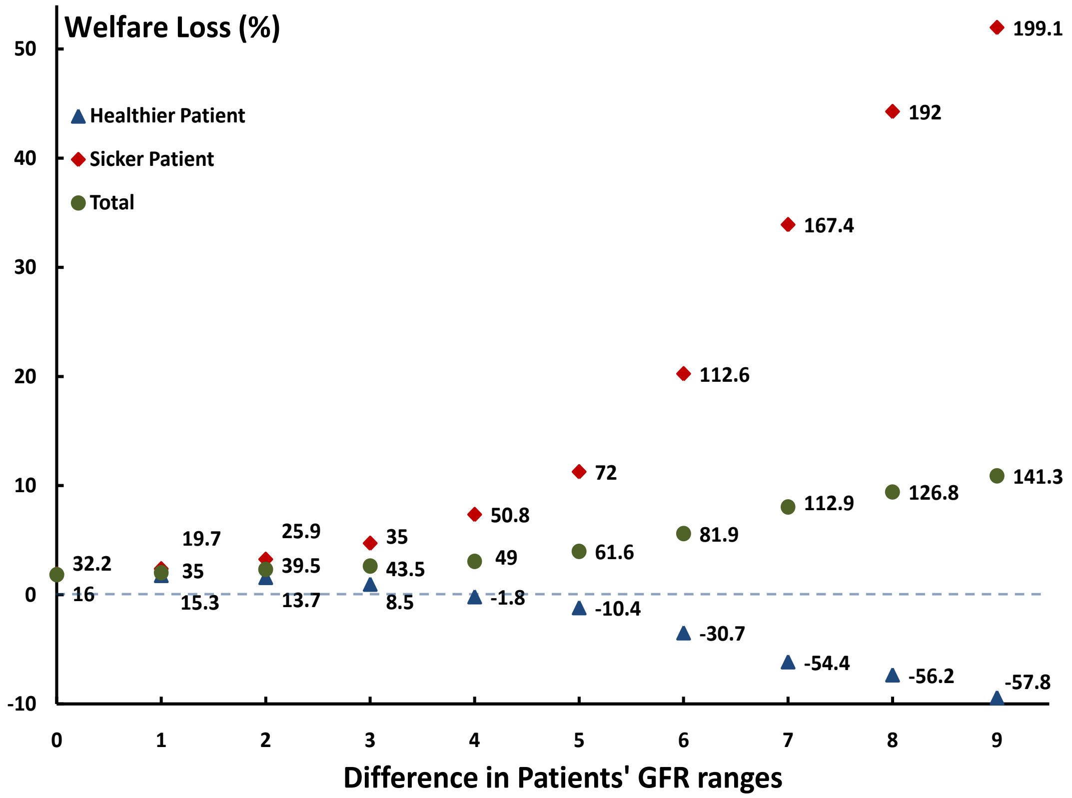

Figure 3: Social welfare loss and patients' individual welfare losses due to patient autonomy. The number next to each data label indicates the loss in absolute terms (in quality-adjusted life weeks).

clinically less desirable than pure strategies, we narrow our focus to pure equilibria and quantify the welfare loss borne by patient autonomy. In the rest of this section, we consider a socially optimal pure equilibrium $\mathbf{A}^*$ so that $\vartheta_i(\widehat{\mathbf{s}}) - g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*)$ denotes the welfare loss for Patient $i \in \mathcal{N}$, and $\sum_{i \in \mathcal{N}} \left[\vartheta_i(\widehat{\mathbf{s}}) - g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*)\right]$ denotes the social welfare loss due to patient autonomy. Note that the socially optimal policy may not be an equilibrium.

From Figure 3, social welfare loss does not decrease as the patients' health statuses diverge, which we interpret as follows: Because the central decision-maker acts solely for the society's interest and the death of one patient leaves the other untransplanted, as one of the patients gets sicker, the central decision-maker becomes more likely to recommend exchange as an optimal decision. On the other hand, when patients are autonomous, the healthier patient can force the sicker patient to wait, although the sicker patient's death would render the exchange infeasible. Also, the sicker patient benefits from the central decision-maker's decisions, and the impact of patient autonomy on her welfare is more dramatic in absolute and relative terms.

As the society's interest may conflict with patients' self-interests, a socially optimal equilibrium strategy may not be an optimal equilibrium strategy that a patient can play. Therefore, for each individual patient we calculate the cost of playing the socially optimal equilibrium strategy rather than any other equilibrium strategy. We let ${}^i\mathbf{A}^* = ({}^i\mathbf{a}_i^*, {}^i\mathbf{a}_{-i}^*)$ denote a pure equilibrium that maximizes Patient i's total expected payoff, *i.e.*, ${}^i\mathbf{A}^* \in \arg\max_{\mathbf{A}\in\Pi} g_i(\widehat{\mathbf{s}}, \mathbf{a}_i, \mathbf{a}_{-i})$. Then, $g_i(\widehat{\mathbf{s}}, {}^i\mathbf{a}_i^*, {}^i\mathbf{a}_{-i}^*) - g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*)$ provides an upper bound for Patient i's cost of playing a socially optimal equilibrium strategy rather than any other equilibrium strategy. For each Patient $i \in \mathcal{N}$, by Theorem 3 (*iii*), we calculate $g_i(\widehat{\mathbf{s}}, {}^i\mathbf{a}_i^*, {}^i\mathbf{a}_{-i}^*)$ by solving the mathematical program $\max_{(\mathbf{w},\mathbf{y},\mathbf{z})\in\Upsilon} w_i(\widehat{\mathbf{s}})$.

Table 2: Patients' maximum welfare losses from following a socially optimal equilibrium. Relative losses are in %, and absolute losses are in quality-adjusted life weeks.

		Difference in Patients' GFR Ranges									
Loss	Patient	0	1	2	3	4	5	6	7	8	9
Relative	Healthier	0.76	1.20	1.49	2.11	2.40	3.00	3.54	3.61	3.78	3.82
	Sicker	0.89	0.80	0.71	0.68	0.61	0.58	0.57	0.50	0.26	0.00
Absolute	Healthier	7.43	9.69	12.20	17.08	19.64	24.96	29.73	26.88	28.64	31.13
	Sicker	7.43	6.57	5.75	5.28	6.04	4.15	3.41	2.67	1.26	0.00

From Table 2, as patients' health statuses diverge, a socially optimal equilibrium considers the sicker patient's interests more, that is, playing a socially optimal equilibrium strategy costs less to the sicker patient. Irrespective of the difference in patients' GFR ranges, a socially optimal equilibrium deviates more from the healthier patient's individually optimal equilibrium than it does from the sicker's. In our experiments, the sicker patient under the socially optimal equilibrium is usually very close to her individually optimal equilibrium. Specifically, in 72% of the cases, the sicker patient's individually optimal equilibrium is within 1% of the socially optimal equilibrium, and in 90% of the cases her individually optimal equilibrium is within 3% of the socially optimal equilibrium.

4.3 Valuing Exchanges: An Example

In this section, we elucidate how the patients' quality-adjusted life expectancies from our model can be used to calibrate edge-weights in graphs used to form patient-donor pairs. We create an example graph of patient-donor pairs for which maximizing the total number of transplants may imply adverse welfare outcomes. In Figure 4, we have 4 patient-donor pairs with specified characteristics.

For the set of patient-donor pairs in Figure 4, we have two possible matching scenarios: In Scenario 1, Patient 1 is matched to Donor 2, Patient 2 is matched to Donor 1, Patient 3 is matched to Donor 4, and Patient 4 is matched to Donor 3. In Scenario 2, Patient 2 is matched to Donor 3, Patient 3 is matched to Donor 2, and Patients 1 and 4 remain untransplanted. Although Scenario 1 maximizes the number of transplants, when we compare the social outcomes under such scenarios, we observe the following: In Scenario 1, an immediate exchange yields 53.27 quality-adjusted life years. When the patients behave autonomously in each of the matchings, the socially optimal equilibria yield 50.49 quality-adjusted life years. In Scenario 2, an immediate exchange yields 52.76 quality-adjusted life years. Because Patients 1 and 4 are untransplanted, when Patients 2 and 3 behave autonomously, a socially optimal equilibria yield 54.27 quality-adjusted life years. Note that in this case, because we assume Patients 1 and 4

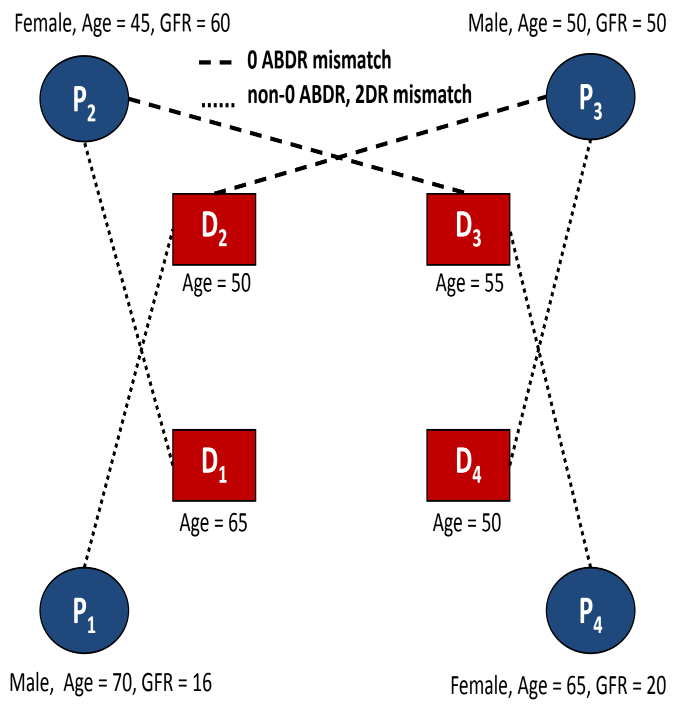

Figure 4: Patient-donor characteristics for the matching example

will never receive a transplant, we calculate their remaining life expectancies by two separate Markov reward chains. Thus, if the patients are autonomous and the edge weight of each possible matching is modeled as the patients' total life expectancies in a socially optimal equilibrium of the game played in that particular matching, then Scenario 2 is optimal although it has fewer transplants than Scenario 1. Note that in Scenario 2, the socially optimal equilibrium provides considerably higher social welfare than that of immediate exchange. Intuitively, when patients behave autonomously, if they are at closer stages of disease, because they conflict less, their matchings yield superior welfare outcomes compared to matchings involving patients at different stages.

5. Highlights

We model the patients' transplant timing decisions in a cyclic PKE as a non-zero-sum stochastic game and analyze the resulting equilibrium selection problem from a social point of view. Our numerical experiments indicate that matching patients based on 0-1 preferences ignoring the timing of the exchange under patient autonomy may result in socially suboptimal circumstances. We also demonstrate that matching patients at similar stages of disease provides more preferable outcomes in terms of the social welfare gained by matching patients based on their life expectancies and the social welfare lost by patient autonomy.

Appendix A: Proofs of Statements

Throughout this appendix, under a strategy profile $\mathbf{A}$, for $\mathbf{s} \in \mathscr{S}$ and $\mathcal{J} \subseteq \mathcal{N}$, we let $Q_{\mathcal{J}}(\mathbf{s}, \mathbf{A}) = \prod_{j \in \mathcal{J}} a_j(\mathbf{s})$ so that $Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A})$ denotes the exchange occurrence probability in state $\mathbf{s} \in \mathscr{S}$ under strategy profile $\mathbf{A}$.

Proof of Theorem 1: ($\Leftarrow$) Consider Patient j. Let $\mathbf{a}'_j = \left[a'_j(\mathbf{s})\right]_{\mathbf{s} \in \mathscr{S}}$ denote a best response of Patient j to the strategy-tuple $\mathbf{a}_{-j}$, implying $\mathbf{g}_j(\mathbf{a}'_j, \mathbf{a}_{-j}) \geq \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$. By definition,

$$g_j(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j}) = a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \left[1 - a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\right] F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \text{ for } \mathbf{s} \in \mathscr{S}.$$

As strategies $\mathbf{a}'_j$ and $\mathbf{a}_{-j}$ are fixed, this recursion defines a stationary, infinite-horizon Markov reward chain. Now, under the strategy profile $\mathbf{A}$, suppose (3) holds for all $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$, and we apply value iteration to this recursion. Let $g_j^n(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j})$ denote the value associated with state $\mathbf{s} \in \mathscr{S}$ at the n^{th} iteration. More specifically, for $n > 0$:

$$g_j^n(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j}) = a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}_j^{n-1}(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \ \text{ for } \mathbf{s} \in \mathscr{S}.$$

By induction on $n \geq 0$, we will show that $\mathbf{g}_j^n(\mathbf{a}'_j, \mathbf{a}_{-j}) \leq \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$ for all $n \geq 0$. Let $\mathbf{g}_j^0(\mathbf{a}'_j, \mathbf{a}_{-j}) = \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$, and for some $m \geq 0$, suppose $\mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j}) \leq \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$, so that $F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \leq F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big)$ for all $\mathbf{s} \in \mathscr{S}$. Now, choose an arbitrary $\mathbf{s} \in \mathscr{S}$ and consider the following possible cases for $g_j^{m+1}(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j})$:

1. If $u_j(\mathbf{s}, 1) \leq F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big)$, then

$$\begin{aligned} g_j^{m+1}(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j}) &= a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \\ &\leq F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \leq F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) \leq g_j(\mathbf{s}, \mathbf{a}_j, \mathbf{a}_{-j}), \end{aligned}$$

   where the last inequality is implied by the assumption that (3) holds for $\mathbf{A}$.

2. If $u_j(\mathbf{s}, 1) > F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big)$, then

$$\begin{aligned} g_j^{m+1}(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j}) &= a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \\ &\leq a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - a'_j(\mathbf{s}) Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \\ &\qquad + Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big[1 - a'_j(\mathbf{s})\big]\big[u_j(\mathbf{s}, 1) - F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big)\big] \qquad \text{(6a)} \\ &= Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}_j^m(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \\ &\leq Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) \leq g_j(\mathbf{s}, \mathbf{a}_j, \mathbf{a}_{-j}), \qquad \text{(6b)} \end{aligned}$$

   where (6a) is implied by the fact that $Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}), a'_j(\mathbf{s}) \in [0, 1]$ and the inequality in (6b) follows from the assumption that (3) holds for all $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$ under $\mathbf{A}$.

Thus, $\mathbf{g}_j^{m+1}(\mathbf{a}'_j, \mathbf{a}_{-j}) \leq \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$. Then, by the convergence of value iteration we obtain $\mathbf{g}_j(\mathbf{a}'_j, \mathbf{a}_{-j}) \leq \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$. Since we already have $\mathbf{g}_j(\mathbf{a}'_j, \mathbf{a}_{-j}) \geq \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$, this implies $\mathbf{g}_j(\mathbf{a}'_j, \mathbf{a}_{-j}) = \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$. Therefore, $\mathbf{a}_j$ is a best response of Patient j to the strategy-tuple $\mathbf{a}_{-j}$.

($\Rightarrow$) The proof is similar to that of ($\Leftarrow$), and omitted. $\square$

Proof of Lemma 1: (i) For Patient j, suppose $g_j(\widehat{\mathbf{s}}, \mathbf{a}_j, \mathbf{a}_{-j}) > u_j(\widehat{\mathbf{s}}, 1)$ for some $\widehat{\mathbf{s}} \in \mathscr{S}$. By (1), $g_j(\widehat{\mathbf{s}}, \mathbf{a}_j, \mathbf{a}_{-j}) \leq \max\left\{u_j(\widehat{\mathbf{s}}, 1), F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big)\right\}$. Then, since $g_j(\widehat{\mathbf{s}}, \mathbf{a}_j, \mathbf{a}_{-j}) > u_j(\widehat{\mathbf{s}}, 1)$, we must have $F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) > u_j(\widehat{\mathbf{s}}, 1)$. Since $\mathbf{A} \in \Gamma$, by Theorem 1, (3) is satisfied for $(\mathbf{s}, i) = (\widehat{\mathbf{s}}, j)$. These imply $g_j(\widehat{\mathbf{s}}, \mathbf{a}_j, \mathbf{a}_{-j}) = F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) > u_j(\widehat{\mathbf{s}}, 1)$. Therefore, by (1), we must have $Q_{\mathcal{N}}(\widehat{\mathbf{s}}, \mathbf{A}) = 0$.

(ii) First, we will show that for any $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$:

$$\text{If } u_i(\mathbf{s}, 1) > F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})\big) \text{ and } Q_{\mathcal{N}_{-i}}(\mathbf{s}, \mathbf{A}) > 0, \text{ then } a_i(\mathbf{s}) = 1. \tag{7}$$

Consider Patient j. For some $\widehat{\mathbf{s}} \in \mathscr{S}$ with $u_j(\widehat{\mathbf{s}}, 1) > F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big)$ and $Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) > 0$ suppose that $a_j(\widehat{\mathbf{s}}) < 1$. While Patients $k \in \mathcal{N}_{-j}$ maintain their strategies, *i.e.*, $\mathbf{a}_{-j}$ is fixed, suppose Patient j follows strategy $\mathbf{a}'_j = \left[a'_j(\mathbf{s})\right]_{\mathbf{s} \in \mathscr{S}}$, where $a'_j(\widehat{\mathbf{s}}) = 1$ and $a'_j(\mathbf{s}) = a_j(\mathbf{s})$ for $\mathbf{s} \in \mathscr{S} \setminus \{\widehat{\mathbf{s}}\}$, so that the payoffs $\mathbf{g}_j(\mathbf{a}'_j, \mathbf{a}_{-j})$ are defined as follows:

$$g_j(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j}) = \begin{cases} Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}'_j, \mathbf{a}_{-j})\big) & \text{for } \mathbf{s} = \widehat{\mathbf{s}}, \\ Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}'_j, \mathbf{a}_{-j})\big) & \text{for } \mathbf{s} \in \mathscr{S} \setminus \{\widehat{\mathbf{s}}\}. \end{cases}$$

As strategies $\mathbf{a}'_j$ and $\mathbf{a}_{-j}$ are fixed, this recursion defines a stationary, infinite-horizon Markov reward chain. Suppose we apply value iteration to this recursion. Let $g^n_j(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j})$ denote the value associated with state $\mathbf{s} \in \mathscr{S}$ at the n^{th} iteration. That is, for $n > 0$:

$$g^n_j(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j}) = \begin{cases} Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}^{n-1}_j(\mathbf{a}'_j, \mathbf{a}_{-j})\big) & \text{for } \mathbf{s} = \widehat{\mathbf{s}}, \\ Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A}) u_j(\mathbf{s}, 1) + \big[1 - Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A})\big] F_j\big(\mathbf{s}, \mathbf{g}^{n-1}_j(\mathbf{a}'_j, \mathbf{a}_{-j})\big) & \text{for } \mathbf{s} \in \mathscr{S} \setminus \{\widehat{\mathbf{s}}\}. \end{cases}$$

By induction on $n \geq 0$, we will show that the following hold for all $n \geq 0$.

$$\mathbf{g}^n_j(\mathbf{a}'_j, \mathbf{a}_{-j}) \geq \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j}) \;\text{ and }\; g^n_j(\widehat{\mathbf{s}}, \mathbf{a}'_j, \mathbf{a}_{-j}) > g_j(\widehat{\mathbf{s}}, \mathbf{a}_j, \mathbf{a}_{-j}). \tag{8}$$

For some finite $\boldsymbol{\epsilon} > \mathbf{0}$, let $\mathbf{g}^0_j(\mathbf{a}'_j, \mathbf{a}_{-j}) = \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j}) + \boldsymbol{\epsilon}$ and suppose (8) holds for some $n = m \geq 0$, so that $F_j\big(\mathbf{s}, \mathbf{g}^m_j(\mathbf{a}'_j, \mathbf{a}_{-j})\big) \geq F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big)$ for all $\mathbf{s} \in \mathscr{S}$. By the definitions of the payoffs $\mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$ and $\mathbf{g}^{m+1}_j(\mathbf{a}'_j, \mathbf{a}_{-j})$, this yields $g^{m+1}_j(\mathbf{s}, \mathbf{a}'_j, \mathbf{a}_{-j}) \geq g_j(\mathbf{s}, \mathbf{a}_j, \mathbf{a}_{-j})$ for all $\mathbf{s} \in \mathscr{S} \setminus \{\widehat{\mathbf{s}}\}$. Next, consider

$g_j^{m+1}(\widehat{\mathbf{s}}, \mathbf{a}_j', \mathbf{a}_{-j})$.

$$\begin{aligned} g_j^{m+1}(\widehat{\mathbf{s}}, \mathbf{a}_j', \mathbf{a}_{-j}) &= Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) u_j(\widehat{\mathbf{s}}, 1) + \big[1 - Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A})\big] F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j^m(\mathbf{a}_j', \mathbf{a}_{-j})\big) \\ &\geq Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) u_j(\widehat{\mathbf{s}}, 1) + \big[1 - Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A})\big] F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) \\ &= a_j(\widehat{\mathbf{s}}) Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) u_j(\widehat{\mathbf{s}}, 1) + \big[1 - a_j(\widehat{\mathbf{s}})\big] Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) u_j(\widehat{\mathbf{s}}, 1) \\ &\qquad + \big[1 - Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A})\big] F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big). \end{aligned} \tag{9}$$

Since $u_j(\widehat{\mathbf{s}}, 1) > F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big)$, $a_j(\widehat{\mathbf{s}}) < 1$ and $Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) > 0$, we have

$$\big[1 - a_j(\widehat{\mathbf{s}})\big] Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) u_j(\widehat{\mathbf{s}}, 1) > \big[1 - a_j(\widehat{\mathbf{s}})\big] Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big).$$

By (9) this implies:

$$\begin{aligned} &g_j^{m+1}(\widehat{\mathbf{s}}, \mathbf{a}_j', \mathbf{a}_{-j}) \\ &\quad > a_j(\widehat{\mathbf{s}}) Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) u_j(\widehat{\mathbf{s}}, 1) + \Big(\big[1 - a_j(\widehat{\mathbf{s}})\big] Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) + \big[1 - Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A})\big]\Big) F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) \\ &\quad = a_j(\widehat{\mathbf{s}}) Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A}) u_j(\widehat{\mathbf{s}}, 1) + \big[1 - a_j(\widehat{\mathbf{s}}) Q_{\mathcal{N}_{-j}}(\widehat{\mathbf{s}}, \mathbf{A})\big] F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) = g_j(\widehat{\mathbf{s}}, \mathbf{a}_j, \mathbf{a}_{-j}). \end{aligned}$$

Thus, (8) holds for $n = m + 1$. Then, the convergence of value iteration implies $\mathbf{g}_j(\mathbf{a}_j', \mathbf{a}_{-j}) \geq \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$ and $g_j(\widehat{\mathbf{s}}, \mathbf{a}_j', \mathbf{a}_{-j}) > g_j(\widehat{\mathbf{s}}, \mathbf{a}_j, \mathbf{a}_{-j})$, which contradicts the assumption that $\mathbf{A} \in \Gamma$. Therefore, $a_j(\widehat{\mathbf{s}}) = 1$ and (7) holds for the pair $(\widehat{\mathbf{s}}, j)$.

Next, for some $\widehat{\mathbf{s}} \in \mathscr{S}$, suppose $Q_{\mathcal{N}}(\widehat{\mathbf{s}}, \mathbf{A}) \in (0, 1)$ implying the existence of a Patient k, where $a_k(\widehat{\mathbf{s}}) \in (0, 1)$. Since $Q_{\mathcal{N}}(\widehat{\mathbf{s}}, \mathbf{A}) > 0$, by part (i), $g_k(\widehat{\mathbf{s}}, \mathbf{a}_k, \mathbf{a}_{-k}) \leq u_k(\widehat{\mathbf{s}}, 1)$. Also, since $Q_{\mathcal{N}_{-k}}(\widehat{\mathbf{s}}, \mathbf{A}) > 0$, $a_k(\widehat{\mathbf{s}}) < 1$ and (7) holds for any $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$, we also have $g_k(\widehat{\mathbf{s}}, \mathbf{a}_k, \mathbf{a}_{-k}) \geq u_k(\widehat{\mathbf{s}}, 1)$. These yield $g_k(\widehat{\mathbf{s}}, \mathbf{a}_k, \mathbf{a}_{-k}) = u_k(\widehat{\mathbf{s}}, 1)$. Since $Q_{\mathcal{N}}(\widehat{\mathbf{s}}, \mathbf{A}) > 0$, by (1), this implies $u_k(\widehat{\mathbf{s}}, 1) = F_k\big(\widehat{\mathbf{s}}, \mathbf{g}_k(\mathbf{a}_k, \mathbf{a}_{-k})\big)$.

(iii) Let $\mathscr{K} = \big\{\mathbf{s} \in \mathscr{S} | Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A}) \in \{0, 1\}\big\}$. Note that $\mathscr{Y}_{\mathbf{s}}(\mathbf{A})$ is nonempty for all $\mathbf{s} \in \mathscr{S} \setminus \mathscr{K}$. Therefore, for $\mathbf{s} \in \mathscr{S} \setminus \mathscr{K}$, let $\eta(\mathbf{s}) = \min\big\{i | i \in \mathscr{Y}_{\mathbf{s}}(\mathbf{A})\big\}$. Now, consider the strategy profile $\mathbf{A}'$ defined by:

$$a'_i(\mathbf{s}) = \begin{cases} a_i(\mathbf{s}) & \text{for } \mathbf{s} \in \mathscr{K} \text{ and } i \in \mathcal{N}, \\ 1 & \text{for } \mathbf{s} \in \mathscr{S} \setminus \mathscr{K} \text{ and } i \neq \eta(\mathbf{s}), \\ Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A}) & \text{for } \mathbf{s} \in \mathscr{S} \setminus \mathscr{K} \text{ and } i = \eta(\mathbf{s}). \end{cases} \tag{10}$$

By the definition of $\mathscr{K}$, (10) implies that $Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A}) = Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A}')$ for all $\mathbf{s} \in \mathscr{S}$. Therefore, by (1),

$$\mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i}) = \mathbf{g}_i(\mathbf{a}'_i, \mathbf{a}'_{-i}) \text{ for all } i \in \mathcal{N}, \tag{11}$$

implying that

$$F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})\big) = F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}'_i, \mathbf{a}'_{-i})\big) \text{ for all } \mathbf{s} \in \mathscr{S} \text{ and } i \in \mathcal{N}. \tag{12}$$

Next, we will show that $\mathbf{A}' \in \Gamma$. By the definition of $\mathscr{K}$, (10) implies that $Q_{\mathcal{N}_{-i}}(\mathbf{s}, \mathbf{A}) = Q_{\mathcal{N}_{-i}}(\mathbf{s}, \mathbf{A}')$ for all $\mathbf{s} \in \mathscr{K}$ and $i \in \mathcal{N}$. Since $\mathbf{A} \in \Gamma$, by Theorem 1, (3) is satisfied for all $\mathbf{s} \in \mathscr{K}$ and $i \in \mathcal{N}$ under $\mathbf{A}$. Therefore, by (11) and (12), (3) is satisfied for all $\mathbf{s} \in \mathscr{K}$ and $i \in \mathcal{N}$ under $\mathbf{A}'$. Now, for an arbitrary $\mathbf{s} \in \mathscr{S} \setminus \mathscr{K}$ and $i \in \mathcal{N}$, consider the following possible cases for $a_i(\mathbf{s})$.

1. If $a_i(\mathbf{s}) \in (0, 1)$, then by part (ii), (1) implies $g_i(\mathbf{s}, \mathbf{a}_i, \mathbf{a}_{-i}) = u_i(\mathbf{s}, 1) = F_i\big(\mathbf{s}, \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i})\big)$. Therefore, by (11) and (12), (3) is satisfied for pair $(\mathbf{s}, i)$ under $\mathbf{A}'$.

2. If $a_i(\mathbf{s}) = 1$, then because $a'_i(\mathbf{s}) = 1$ and $Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A}) = Q_{\mathcal{N}}(\mathbf{s}, \mathbf{A}')$, we have $Q_{\mathcal{N}_{-i}}(\mathbf{s}, \mathbf{A}) = Q_{\mathcal{N}_{-i}}(\mathbf{s}, \mathbf{A}')$. Since $\mathbf{A} \in \Gamma$, by (11) and (12), this implies that (3) is satisfied for pair $(\mathbf{s}, i)$ under $\mathbf{A}'$.

Thus, (3) is satisfied for all $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$ under $\mathbf{A}'$. Therefore, by Theorem 1, $\mathbf{A}' \in \Gamma$. By the definition of $\mathscr{K}$ and (10), it is easy to see that $|\mathscr{Y}_{\mathbf{s}}(\mathbf{A}')| \leq 1$ for all $\mathbf{s} \in \mathscr{S}$. $\square$

Proof of Lemma 2: By (1) and the definition of $\mathbf{V}_i$, the proof follows by induction on the iterates of value iteration, and is omitted. $\square$

Proof of Theorem 2: (i) Given $\widehat{\mathbf{A}} \in \Gamma$, let $\mathscr{B}_\ell = \big\{\mathbf{s} \in \mathscr{S} | Q_{\mathcal{N}}(\mathbf{s}, \widehat{\mathbf{A}}) = \ell\big\}$ for $\ell \in \{0, 1\}$ and $\mathscr{C} = \mathscr{S} \setminus (\mathscr{B}_0 \cup \mathscr{B}_1)$. By the definition of $\mathscr{C}$, Lemma 1 (ii) implies that for each $\mathbf{s} \in \mathscr{C}$, $\exists\, i \in \mathcal{N}$ with $u_i(\mathbf{s}, 1) = F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big)$. Therefore, for $\mathbf{s} \in \mathscr{C}$, let $\mathscr{H}(\mathbf{s}) = \min\big\{i \in \mathcal{N} | u_i(\mathbf{s}, 1) = F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big)\big\}$. Now, we will construct a solution $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ satisfying constraints (4a)-(4j).

- For $\ell \in \{0,1\}$, $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$, let $\widehat{y}_i(\mathbf{s}) = \ell$ if $\mathbf{s} \in \mathscr{B}_\ell$.
- For $\mathbf{s} \in \mathscr{C}$, let $\widehat{y}_i(\mathbf{s}) = 0$ if $i = \mathscr{H}(\mathbf{s})$, and $\widehat{y}_i(\mathbf{s}) = 1$ otherwise.
- For $\mathbf{s} \in \mathscr{S}$, let $\widehat{z}(\mathbf{s}) = 0$ if $\mathbf{s} \in \mathscr{B}_0$, and $\widehat{z}(\mathbf{s}) = 1$ otherwise.
- For $i \in \mathcal{N}$, let

$$\widehat{\mathbf{w}}_i = \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}). \tag{13}$$

As an immediate consequence of assignment (13):

$$F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big) = F_i(\mathbf{s}, \widehat{\mathbf{w}}_i) \text{ for all } \mathbf{s} \in \mathscr{S} \text{ and } i \in \mathcal{N}. \tag{14}$$

By the construction of $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{z}}$, constraints (4f)-(4i) are satisfied. Because $\widehat{\mathbf{A}} \in \Gamma$, by Theorem 1, $g_i(\mathbf{s}, \widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) \geq F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big)$ for all $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$. By (13) and (14), this implies that constraints (4a) are satisfied. Since $\mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) \geq \mathbf{0}$ for all $i \in \mathcal{N}$, (13) also imply that $\widehat{\mathbf{w}}_i \geq \mathbf{0}$ for all $i \in \mathcal{N}$. Therefore, constraints (4j) are satisfied. Next, we will show that $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ satisfy constraints (4b)-(4e). Consider the following possible cases for an arbitrary $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$:

1. If $\mathbf{s} \in \mathscr{B}_0$, then since $\widehat{y}_j(\mathbf{s}) = 0$ for all $j \in \mathcal{N}$ and $\widehat{\mathbf{w}}_i \geq \mathbf{0}$, constraints (4c) and (4d) are satisfied. Since $\widehat{z}(\mathbf{s}) = 0$ and $g_i(\mathbf{s}, \widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) \leq V_i(\mathbf{s})$ by Lemma 2, constraint (4e) is satisfied by (13). Since $Q_{\mathcal{N}}(\mathbf{s}, \widehat{\mathbf{A}}) = 0$, by (1), $g_i(\mathbf{s}, \widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) = F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big)$. Then, since $\widehat{y}_i(\mathbf{s}) = 0$, (13) and (14) imply that constraint (4b) is satisfied.

2. If $\mathbf{s} \in \mathscr{B}_1$, then since $Q_{\mathcal{N}}(\mathbf{s}, \widehat{\mathbf{A}}) = 1$, by (1) $g_i(\mathbf{s}, \widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) = u_i(\mathbf{s}, 1)$. Therefore, by (13), $\widehat{w}_i(\mathbf{s}) = u_i(\mathbf{s}, 1)$. Then, since $\widehat{y}_i(\mathbf{s}) = 1$ and $F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big) \geq 0$, (14) implies that constraint (4b) is satisfied. Because $\widehat{z}(\mathbf{s}) = 1$ and $\widehat{y}_j(\mathbf{s}) = 1$ for all $j \in \mathcal{N}$, constraints (4d) and (4e) are also satisfied by $\widehat{w}_i(\mathbf{s}) = u_i(\mathbf{s}, 1)$. Finally, because $\widehat{y}_j(\mathbf{s}) = 1$ for all $j \in \mathcal{N}$, constraint (4c) is satisfied by the nonnegativity of $\widehat{w}_i(\mathbf{s})$.

3. If $\mathbf{s} \in \mathscr{C}$, then since $Q_{\mathcal{N}}(\mathbf{s}, \widehat{\mathbf{A}}) \in (0,1)$, by Lemma 1 (i), $g_i(\mathbf{s}, \widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) \leq u_i(\mathbf{s}, 1)$. Therefore, by (13), $\widehat{w}_i(\mathbf{s}) \leq u_i(\mathbf{s}, 1)$. By the definition of the value function $\mathbf{V}_i$, $V_i(\mathbf{s}) \geq u_i(\mathbf{s}, 1)$. Therefore,

constraint (4e) is satisfied. Also, by the construction of $\widehat{\mathbf{y}}$, $\sum_{j\in\mathcal{N}} \widehat{y}_j(\mathbf{s}) = N-1$. Since $\widehat{w}_i(\mathbf{s}) \geq 0$, this implies that constraint (4d) is satisfied. Now, consider the two possible cases for i.

(a) If $i \neq \mathscr{H}(\mathbf{s})$, then because $\widehat{y}_i(\mathbf{s}) = 1$, $\widehat{w}_i(\mathbf{s}) \leq u_i(\mathbf{s},1)$ and $F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big) \geq 0$, (14) implies that constraint (4b) is satisfied. By the construction of $\widehat{\mathbf{y}}$, $\widehat{y}_j(\mathbf{s}) - \widehat{y}_i(\mathbf{s}) \leq 0$ for all $j \in \mathcal{N}_{-i}$. Then, since $\widehat{w}_i(\mathbf{s}) \geq 0$, constraint (4c) is satisfied.

(b) If $i = \mathscr{H}(\mathbf{s})$, then since $u_i(\mathbf{s},1) = F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big)$, by (1),

$$g_i(\mathbf{s}, \widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) = u_i(\mathbf{s},1) = F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big). \tag{15}$$

Since $\widehat{y}_i(\mathbf{s}) = 0$, by (13), (14) and (15), constraint (4b) is satisfied. Furthermore, because $\widehat{y}_j(\mathbf{s}) = 1$ for $j \in \mathcal{N}_{-i}$, by (13) and (15) constraint (4c) is satisfied.

Thus, $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) \in \Lambda$.

(ii) The result immediately follows from part *(i)*, and the proof is omitted.

(iii) For $\ell \in \{0,1\}$, let $\mathscr{W}_\ell = \big\{\mathbf{s} \in \mathscr{S} | \widehat{y}_1(\mathbf{s}) = \widehat{y}_2(\mathbf{s}) = \ell\big\}$ and for $i \in \mathcal{N}$ let $\mathscr{R}_i = \big\{\mathbf{s} \in \mathscr{S} | \widehat{y}_i(\mathbf{s}) = 0, \widehat{y}_{-i}(\mathbf{s}) = 1\big\}$ and $\mathscr{Z}_i = \big\{\mathbf{s} \in \mathscr{R}_i | u_{-i}(\mathbf{s},1) = F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i})\big\}$. Note that constraints (4a) hold, but in particular:

$$\text{For any } i \in \mathcal{N}\text{: } \widehat{w}_{-i}(\mathbf{s}) \geq F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i}) \text{ for all } \mathbf{s} \in \mathscr{R}_i. \tag{16a}$$

Also by constraints (4e), (4f) and (4i) for any $i \in \mathcal{N}$, $u_{-i}(\mathbf{s},1) \geq w_{-i}(\mathbf{s})$ for all $\mathbf{s} \in \mathscr{R}_i$. By (16a) and the definition of $\mathscr{Z}_i$ for $i \in \mathcal{N}$, this implies:

$$\text{For any } i \in \mathcal{N}\text{: } u_{-i}(\mathbf{s},1) > F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i}) \text{ for all } \mathbf{s} \in \mathscr{R}_i \setminus \mathscr{Z}_i. \tag{16b}$$

Thus, by (16a) and (16b), for any $i \in \mathcal{N}$, $\left(\dfrac{\widehat{w}_{-i}(\mathbf{s}) - F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i})}{u_{-i}(\mathbf{s},1) - F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i})}\right) \in [0,1]$ for all $\mathbf{s} \in \mathscr{R}_i \setminus \mathscr{Z}_i$. Now, consider the strategy profile $\widehat{\mathbf{A}}$ defined by:

$$\widehat{a}_i(\mathbf{s}) = \begin{cases} 0 & \text{for } \mathbf{s} \in \mathscr{W}_0 \cup \mathscr{Z}_i, \\ 1 & \text{for } \mathbf{s} \in \mathscr{W}_1 \cup \mathscr{R}_{-i}, \\ \dfrac{\widehat{w}_{-i}(\mathbf{s}) - F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i})}{u_{-i}(\mathbf{s},1) - F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i})} & \text{for } \mathbf{s} \in \mathscr{R}_i \setminus \mathscr{Z}_i. \end{cases} \tag{17}$$

By (17), for the product $\widehat{a}_1(\mathbf{s})\widehat{a}_2(\mathbf{s})$ we have:

$$\widehat{a}_1(\mathbf{s})\widehat{a}_2(\mathbf{s}) = \begin{cases} 0 & \text{for } \mathbf{s} \in \mathscr{W}_0 \cup \mathscr{Z}_1 \cup \mathscr{Z}_2, \\ 1 & \text{for } \mathbf{s} \in \mathscr{W}_1, \\ \dfrac{\widehat{w}_{-i}(\mathbf{s}) - F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i})}{u_{-i}(\mathbf{s},1) - F_{-i}(\mathbf{s}, \widehat{\mathbf{w}}_{-i})} & \text{for } \mathbf{s} \in \mathscr{R}_i \setminus \mathscr{Z}_i,\ i \in \mathcal{N}. \end{cases} \tag{18}$$

Consider Patient 1. First, we will show that $\widehat{\mathbf{w}}_1 = \mathbf{g}_1(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)$. By (18), (1) defines the payoffs $\mathbf{g}_1(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)$ as:

$$g_1(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \begin{cases} F_1\big(\mathbf{s}, \mathbf{g}_1(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) & \text{for } \mathbf{s} \in \mathscr{W}_0 \cup \mathscr{Z}_1 \cup \mathscr{Z}_2, \\ u_1(\mathbf{s},1) & \text{for } \mathbf{s} \in \mathscr{W}_1, \\ \left(\dfrac{\widehat{w}_2(\mathbf{s}) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}{u_2(\mathbf{s},1) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}\right) u_1(\mathbf{s},1) + \left(\dfrac{u_2(\mathbf{s},1) - \widehat{w}_2(\mathbf{s})}{u_2(\mathbf{s},1) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}\right) F_1\big(\mathbf{s}, \mathbf{g}_1(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) & \\ \qquad \text{for } \mathbf{s} \in \mathscr{R}_1 \setminus \mathscr{Z}_1, & \\ \left(\dfrac{\widehat{w}_1(\mathbf{s}) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}{u_1(\mathbf{s},1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) u_1(\mathbf{s},1) + \left(\dfrac{u_1(\mathbf{s},1) - \widehat{w}_1(\mathbf{s})}{u_1(\mathbf{s},1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) F_1\big(\mathbf{s}, \mathbf{g}_1(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) & \\ \qquad \text{for } \mathbf{s} \in \mathscr{R}_2 \setminus \mathscr{Z}_2. & \end{cases} \tag{19}$$

Because $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) \in \Lambda$, by constraints (4a), (4b), (4e), (4f) and (4i), and the definition of $\mathscr{Z}_2$:

$$\widehat{w}_1(\mathbf{s}) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1) \ \text{ for all } \mathbf{s} \in \mathscr{W}_0 \cup \mathscr{Z}_2. \tag{20a}$$

By constraints (4a), (4d), (4e), (4f) and (4i):

$$\widehat{w}_1(\mathbf{s}) = u_1(\mathbf{s},1) \geq F_1(\mathbf{s}, \widehat{\mathbf{w}}_1) \ \text{ for all } \mathbf{s} \in \mathscr{W}_1. \tag{20b}$$

Also, by constraints (4a), (4b), (4c), (4e), (4f) and (4i):

$$\widehat{w}_1(\mathbf{s}) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1) = u_1(\mathbf{s},1) \ \text{ for all } \mathbf{s} \in \mathscr{R}_1. \tag{20c}$$

Note that as strategies $\widehat{\mathbf{a}}_1$ and $\widehat{\mathbf{a}}_2$ are fixed, recursion (19) defines a stationary, infinite-horizon Markov reward chain. Suppose we apply value iteration to this recursion. Let $g_1^n(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)$ denote the value associated with state $\mathbf{s} \in \mathscr{S}$ at the n^{th} iteration. More explicitly, for $n > 0$:

$$g_1^n(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \begin{cases} F_1\big(\mathbf{s}, \mathbf{g}_1^{n-1}(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) & \text{for } \mathbf{s} \in \mathscr{W}_0 \cup \mathscr{Z}_1 \cup \mathscr{Z}_2, \\ u_1(\mathbf{s}, 1) & \text{for } \mathbf{s} \in \mathscr{W}_1, \\ \left(\dfrac{\widehat{w}_2(\mathbf{s}) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}{u_2(\mathbf{s}, 1) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}\right) u_1(\mathbf{s}, 1) + \left(\dfrac{u_2(\mathbf{s}, 1) - \widehat{w}_2(\mathbf{s})}{u_2(\mathbf{s}, 1) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}\right) F_1\big(\mathbf{s}, \mathbf{g}_1^{n-1}(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) & \\ \qquad \text{for } \mathbf{s} \in \mathscr{R}_1 \setminus \mathscr{Z}_1, & \\ \left(\dfrac{\widehat{w}_1(\mathbf{s}) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}{u_1(\mathbf{s}, 1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) u_1(\mathbf{s}, 1) + \left(\dfrac{u_1(\mathbf{s}, 1) - \widehat{w}_1(\mathbf{s})}{u_1(\mathbf{s}, 1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) F_1\big(\mathbf{s}, \mathbf{g}_1^{n-1}(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) & \\ \qquad \text{for } \mathbf{s} \in \mathscr{R}_2 \setminus \mathscr{Z}_2. & \end{cases}$$

By induction on $n \geq 0$, we will show that $\mathbf{g}_1^n(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \widehat{\mathbf{w}}_1$ for all $n \geq 0$. Let $\mathbf{g}_1^0(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \widehat{\mathbf{w}}_1$, and for some $m \geq 0$, suppose $\mathbf{g}_1^m(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \widehat{\mathbf{w}}_1$. This implies $F_1\big(\mathbf{s}, \mathbf{g}_1^m(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)$ for all $\mathbf{s} \in \mathscr{S}$. Next, choose an arbitrary $\mathbf{s} \in \mathscr{S}$ and consider the following possible cases for $g_1^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)$.

1. If $\mathbf{s} \in \mathscr{W}_0 \cup \mathscr{Z}_1 \cup \mathscr{Z}_2$, then since $\mathscr{Z}_1 \subseteq \mathscr{R}_1$ and $F_1\big(\mathbf{s}, \mathbf{g}_1^m(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)$, by the definition of $g_1^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)$, (20a) and (20c) imply $g_1^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \widehat{w}_1(\mathbf{s})$.

2. If $\mathbf{s} \in \mathscr{W}_1$, then by the definition of $g_1^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)$, (20b) implies $g_1^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \widehat{w}_1(\mathbf{s})$.

3. If $\mathbf{s} \in \mathscr{R}_1 \setminus \mathscr{Z}_1$, then

$$\begin{aligned} g_1^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) &= \left(\frac{\widehat{w}_2(\mathbf{s}) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}{u_2(\mathbf{s}, 1) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}\right) u_1(\mathbf{s}, 1) + \left(\frac{u_2(\mathbf{s}, 1) - \widehat{w}_2(\mathbf{s})}{u_2(\mathbf{s}, 1) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}\right) F_1\big(\mathbf{s}, \mathbf{g}_1^m(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) \\ &= \left(\frac{\widehat{w}_2(\mathbf{s}) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}{u_2(\mathbf{s}, 1) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}\right) u_1(\mathbf{s}, 1) + \left(\frac{u_2(\mathbf{s}, 1) - \widehat{w}_2(\mathbf{s})}{u_2(\mathbf{s}, 1) - F_2(\mathbf{s}, \widehat{\mathbf{w}}_2)}\right) F_1(\mathbf{s}, \widehat{\mathbf{w}}_1) = \widehat{w}_1(\mathbf{s}), \end{aligned} \tag{21}$$

   where the first equality in (21) is implied by $F_1\big(\mathbf{s}, \mathbf{g}_1^m(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)$, and the second equality follows from (20c).

4. If $\mathbf{s} \in \mathscr{R}_2 \setminus \mathscr{Z}_2$, then

$$\begin{aligned} g_1^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) &= \left(\frac{\widehat{w}_1(\mathbf{s}) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}{u_1(\mathbf{s}, 1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) u_1(\mathbf{s}, 1) + \left(\frac{u_1(\mathbf{s}, 1) - \widehat{w}_1(\mathbf{s})}{u_1(\mathbf{s}, 1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) F_1\big(\mathbf{s}, \mathbf{g}_1^m(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) \\ &= \left(\frac{\widehat{w}_1(\mathbf{s}) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}{u_1(\mathbf{s}, 1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) u_1(\mathbf{s}, 1) + \left(\frac{u_1(\mathbf{s}, 1) - \widehat{w}_1(\mathbf{s})}{u_1(\mathbf{s}, 1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) F_1(\mathbf{s}, \widehat{\mathbf{w}}_1) = \widehat{w}_1(\mathbf{s}), \end{aligned} \tag{22}$$

   where the first equality in (22) is implied by $F_1\big(\mathbf{s}, \mathbf{g}_1^m(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)$.

Thus, $\mathbf{g}_1^{m+1}(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \widehat{\mathbf{w}}_1$. Then, the convergence of value iteration implies $\mathbf{g}_1(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \widehat{\mathbf{w}}_1$. Next, we will prove that $\widehat{\mathbf{A}} \in \Gamma$. Since $\mathbf{g}_1(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2) = \widehat{\mathbf{w}}_1$ and $F_1\big(\mathbf{s}, \mathbf{g}_1(\widehat{\mathbf{a}}_1, \widehat{\mathbf{a}}_2)\big) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)$ for $\mathbf{s} \in \mathscr{S}$, by Theorem 1, it is sufficient to show that the following holds for all $\mathbf{s} \in \mathscr{S}$.

$$\widehat{w}_1(\mathbf{s}) = \max\big\{\widehat{a}_2(\mathbf{s})u_1(\mathbf{s},1) + \big[1 - \widehat{a}_2(\mathbf{s})\big]F_1(\mathbf{s}, \widehat{\mathbf{w}}_1), F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)\big\}. \tag{23}$$

Choose an arbitrary $\mathbf{s} \in \mathscr{S}$ and consider the following possible cases for $\widehat{w}_1(\mathbf{s})$:

1. If $\mathbf{s} \in \mathscr{W}_0$, then by (20a), $\widehat{w}_1(\mathbf{s}) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)$. Then, since $\widehat{a}_2(\mathbf{s}) = 0$, (23) is satisfied.

2. If $\mathbf{s} \in \mathscr{W}_1 \cup \mathscr{R}_1$, then by (20b) and (20c), $\widehat{w}_1(\mathbf{s}) = \max\big\{u_1(\mathbf{s},1), F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)\big\}$. Then, since $\widehat{a}_2(\mathbf{s}) = 1$, (23) holds.

3. If $\mathbf{s} \in \mathscr{Z}_2$, then since $u_1(\mathbf{s},1) = F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)$, (23) is satisfied.

4. If $\mathbf{s} \in \mathscr{R}_2 \setminus \mathscr{Z}_2$, then since $\widehat{a}_2(\mathbf{s}) = \left(\dfrac{\widehat{w}_1(\mathbf{s}) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}{u_1(\mathbf{s},1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right)$ by (17), we have:

$$\begin{aligned}\widehat{w}_1(\mathbf{s}) &= \left(\frac{\widehat{w}_1(\mathbf{s}) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}{u_1(\mathbf{s},1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) u_1(\mathbf{s},1) + \left(\frac{u_1(\mathbf{s},1) - \widehat{w}_1(\mathbf{s})}{u_1(\mathbf{s},1) - F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)}\right) F_1(\mathbf{s}, \widehat{\mathbf{w}}_1) \\ &= \widehat{a}_2(\mathbf{s})u_1(\mathbf{s},1) + \big[1 - \widehat{a}_2(\mathbf{s})\big]F_1(\mathbf{s}, \widehat{\mathbf{w}}_1). \end{aligned} \tag{24}$$

   Since $\mathbf{s} \in \mathscr{R}_2 \setminus \mathscr{Z}_2$, by (16b), $u_1(\mathbf{s},1) > F_1(\mathbf{s}, \widehat{\mathbf{w}}_1)$. By (24) and the fact that $\widehat{a}_2(\mathbf{s}) \in [0,1]$, this implies that (23) holds.

(iv) The result immediately follows from parts (i) and (iii), and the proof is omitted. $\square$

Proof of Theorem 3: (i) $(\Leftarrow)$ By Theorem 1, it is sufficient to show that under the strategy profile $\mathbf{A}$ recursion (3) is satisfied for all $\mathbf{s} \in \mathscr{S}$ and $i \in \mathscr{N}$. Consider Patient j. Since $a_i(\mathbf{s}) \in \{0,1\}$ for all $\mathbf{s} \in \mathscr{S}$ and $i \in \mathscr{N}$, by (1), the payoffs $\mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$ are defined as follows:

$$g_j(\mathbf{s}, \mathbf{a}_j, \mathbf{a}_{-j}) = \begin{cases} u_j(\mathbf{s},1) & \text{if } a_k(\mathbf{s}) = 1 \text{ for } k \in \mathscr{N}, \\ F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) & \text{otherwise.} \end{cases} \tag{25}$$

Note that by (5), when $Q_{\mathscr{N}_{-j}}(\mathbf{s}, \mathbf{A}) = 1$:

$$\max\big\{u_j(\mathbf{s},1), F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big)\big\} = \begin{cases} u_j(\mathbf{s},1) & \text{if } a_j(\mathbf{s}) = 1, \\ F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) & \text{if } a_j(\mathbf{s}) = 0. \end{cases} \tag{26}$$

By (25), for Patient j, (26) implies that (3) is satisfied for all $\mathbf{s} \in \mathscr{S}$.

($\Rightarrow$) Consider Patient j. Since $\mathbf{A} \in \Gamma$, by Theorem 1, the payoffs $\mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})$ satisfy the following recursion:

$$g_j(\mathbf{s}, \mathbf{a}_j, \mathbf{a}_{-j}) = \begin{cases} \max\big\{u_j(\mathbf{s}, 1), F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big)\big\} & \text{if } Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) = 1, \\ \\ F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) & \text{otherwise.} \end{cases} \tag{27}$$

Now, choose an arbitrary $\mathbf{s} \in \mathscr{S}$ and for the two possible cases of $Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A})$, consider $a_j(\mathbf{s})$.

1. If $Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) = 0$, then by (1) and (27), we must have

$$F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big) = \big[1 - a_j(\mathbf{s})0\big] F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big).$$

   Therefore, either $a_j(\mathbf{s}) = 0$ or $a_j(\mathbf{s}) = 1$.

2. If $Q_{\mathcal{N}_{-j}}(\mathbf{s}, \mathbf{A}) = 1$, then by (1) and (27), we must have

$$\max\big\{u_j(\mathbf{s}, 1), F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big)\big\} = a_j(\mathbf{s}) u_j(\mathbf{s}, 1) + \big[1 - a_j(\mathbf{s})\big] F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big). \tag{28}$$

   Since $a_j(\mathbf{s}) \in \{0, 1\}$, by (28), it must satisfy the following:

$$a_j(\mathbf{s}) = \begin{cases} 1 & \text{if } u_j(\mathbf{s}, 1) \geq F_j\big(\mathbf{s}, \mathbf{g}_j(\mathbf{a}_j, \mathbf{a}_{-j})\big), \\ \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for Patient j, (5) is satisfied for all $\mathbf{s} \in \mathscr{S}$.

(ii) ($\Leftarrow$) Let $\widehat{\mathbf{a}}_i = \widehat{\mathbf{y}}_i$ for all $i \in \mathcal{N}$. By Theorem 1, it is sufficient to show that recursion (3) is satisfied for all $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$ under the strategy profile $\widehat{\mathbf{A}}$. Consider Patient i. Since $\widehat{\mathbf{y}}_j = \widehat{\mathbf{y}}_{j+1}$ for $j \in \mathcal{N}_{-N}$, $Q_{\mathcal{N}_{-i}}(\mathbf{s}, \widehat{\mathbf{A}}) = \widehat{a}_i(\mathbf{s}) Q_{\mathcal{N}_{-i}}(\mathbf{s}, \widehat{\mathbf{A}})$ for all $\mathbf{s} \in \mathscr{S}$ so that the payoffs $\mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})$ are defined as follows:

$$g_i(\mathbf{s}, \widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) = \begin{cases} F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big) & \text{if } Q_{\mathcal{N}_{-i}}(\mathbf{s}, \widehat{\mathbf{A}}) = 0, \\ \\ u_i(\mathbf{s}, 1) & \text{if } Q_{\mathcal{N}_{-i}}(\mathbf{s}, \widehat{\mathbf{A}}) = 1. \end{cases} \tag{29}$$

Since $\widehat{\mathbf{a}}_j = \widehat{\mathbf{y}}_j$ for all $j \in \mathcal{N}$ and $\widehat{\mathbf{y}}_j = \widehat{\mathbf{y}}_{j+1}$ for $j \in \mathcal{N}_{-N}$, by constraints (4a) and (4b),

$$\widehat{w}_i(\mathbf{s}) = F_i(\mathbf{s}, \widehat{\mathbf{w}}_i) \ \text{ if } Q_{\mathcal{N}_{-i}}(\mathbf{s}, \widehat{\mathbf{A}}) = 0; \tag{30a}$$

and, by constraints (4a), (4d), (4e), (4f) and (4i):

$$\widehat{w}_i(\mathbf{s}) = u_i(\mathbf{s}, 1) \geq F_i(\mathbf{s}, \widehat{\mathbf{w}}_i) \ \text{ if } Q_{\mathcal{N}_{-i}}(\mathbf{s}, \widehat{\mathbf{A}}) = 1. \tag{30b}$$

By (29) - (30b), it is clear that $\mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i}) = \widehat{\mathbf{w}}_i$. Therefore, $F_i\big(\mathbf{s}, \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})\big) = F_i(\mathbf{s}, \widehat{\mathbf{w}}_i)$ for all $\mathbf{s} \in \mathscr{S}$. By (29) and (30b), for Patient i, this implies that (3) is satisfied for all $\mathbf{s} \in \mathscr{S}$.

($\Rightarrow$) Given $\widehat{\mathbf{A}} \in \Pi$, after setting $\widehat{z}(\mathbf{s}) = \widehat{y}_i(\mathbf{s}) = Q_{\mathcal{N}}(\mathbf{s}, \widehat{\mathbf{A}})$ and $\widehat{\mathbf{w}}_i = \mathbf{g}_i(\widehat{\mathbf{a}}_i, \widehat{\mathbf{a}}_{-i})$ for all $i \in \mathcal{N}$, similar to the proof of ($\Leftarrow$) it can be easily shown that $(\widehat{\mathbf{w}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) \in \Upsilon$.

(iii) The result immediately follows from part (ii), and the proof is omitted.

(iv) We will establish the result in three steps. Let $\Pi_{\widehat{\mathbf{s}}} = \big\{\mathbf{A} \in \Pi | Q_{\mathcal{N}}(\widehat{\mathbf{s}}, \mathbf{A}) = 1\big\}$ and for each patient $i \in \mathcal{N}$, define $d_i(\mathbf{s}) = F_i(\mathbf{s}, \mathbf{d}_i)$ for $\mathbf{s} \in \mathscr{S}$. First, we will show that

$$\text{For any } \ \mathbf{A} \in \Pi : \ \ \mathbf{g}_i(\mathbf{a}_i, \mathbf{a}_{-i}) \geq \mathbf{d}_i \ \text{ for all } i \in \mathcal{N}. \tag{31}$$

Given $\widehat{\mathbf{A}} \in \Pi$, consider Patient j. By Theorem 1, for $\mathbf{s} \in \mathscr{S}$:

$$g_j(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) = \max\Big\{Q_{\mathcal{N}_{-j}}(\mathbf{s}, \widehat{\mathbf{A}}) u_j(\mathbf{s}, 1) + \big[1 - Q_{\mathcal{N}_{-j}}(\mathbf{s}, \widehat{\mathbf{A}})\big] F_j\big(\mathbf{s}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big), F_j\big(\mathbf{s}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big)\Big\}.$$

As the strategy-tuple $\widehat{\mathbf{a}}_{-j}$ is fixed, this recursion defines a stationary, infinite-horizon Markov reward chain. Now, suppose we apply value iteration to this recursion. Let $g_j^n(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})$ denote the value associated with state $\mathbf{s} \in \mathscr{S}$ at the n^{th} iteration. More explicitly, for $\mathbf{s} \in \mathscr{S}$ and $n > 0$:

$$\begin{aligned} g_j^n(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) = \max\Big\{ & Q_{\mathcal{N}_{-j}}(\mathbf{s}, \widehat{\mathbf{A}}) u_j(\mathbf{s}, 1) + \big[1 - Q_{\mathcal{N}_{-j}}(\mathbf{s}, \widehat{\mathbf{A}})\big] F_j\big(\mathbf{s}, \mathbf{g}_j^{n-1}(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big), \\ & F_j\big(\mathbf{s}, \mathbf{g}_j^{n-1}(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big)\Big\}. \end{aligned}$$

By induction on $n \geq 0$, we will show that $\mathbf{g}_j^n(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) \geq \mathbf{d}_j$ for all $n \geq 0$. Let $\mathbf{g}_j^0(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) = \mathbf{d}_j$, and for some $m \geq 0$, suppose $\mathbf{g}_j^m(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) \geq \mathbf{d}_j$, so that $F_j\big(\mathbf{s}, \mathbf{g}_j^m(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big) \geq F_j(\mathbf{s}, \mathbf{d}_j)$ for all $\mathbf{s} \in \mathscr{S}$. By the definition of the payoffs $\mathbf{g}_j^{m+1}(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})$ and $\mathbf{d}_j$, we also have $g_j^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) \geq F_j\big(\mathbf{s}, \mathbf{g}_j^m(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big)$ and $d_j(\mathbf{s}) = F_j(\mathbf{s}, \mathbf{d}_j)$ for $\mathbf{s} \in \mathscr{S}$. Therefore, $\mathbf{g}_j^{m+1}(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) \geq \mathbf{d}_j$, and by the convergence of value iteration we obtain $\mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) \geq \mathbf{d}_j$.

Secondly, we will show that if $u_i(\widehat{\mathbf{s}}, 1) \geq d_i(\widehat{\mathbf{s}})$ for all $i \in \mathcal{N}$ then $\Pi_{\widehat{\mathbf{s}}} \neq \emptyset$. Construct the strategy profile $\widehat{\mathbf{A}}$ as follows: For $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$, let $\widehat{a}_i(\mathbf{s}) = 1$ if $\mathbf{s} = \widehat{\mathbf{s}}$, and $\widehat{a}_i(\mathbf{s}) = 0$ otherwise. By Theorem 1, it is sufficient to show that recursion (3) is satisfied for all $\mathbf{s} \in \mathscr{S}$ and $i \in \mathcal{N}$ under $\widehat{\mathbf{A}}$. Consider Patient j. By the construction of $\widehat{\mathbf{A}}$, (1) defines the payoffs $\mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})$ as:

$$g_j(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) = \begin{cases} u_j(\mathbf{s}, 1) & \text{if } \mathbf{s} = \widehat{\mathbf{s}}, \\ F_j\big(\mathbf{s}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big) & \text{otherwise.} \end{cases} \tag{32}$$

Let $\mu = u_j(\widehat{\mathbf{s}}, 1) - d_j(\widehat{\mathbf{s}})$. First, we will show that

$$g_j(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) \in \big[d_j(\mathbf{s}), d_j(\mathbf{s}) + \mu\big] \text{ for all } \mathbf{s} \in \mathscr{S}. \tag{33}$$

As $\widehat{\mathbf{A}}$ is fixed, recursion (32) defines a stationary, infinite-horizon Markov reward chain. Now, suppose we apply value iteration to recursion (32). Let $g_j^n(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})$ denote the value associated with state $\mathbf{s} \in \mathscr{S}$ at the n^{th} iteration. That is, for $n > 0$:

$$g_j^n(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) = \begin{cases} u_j(\mathbf{s}, 1) & \text{if } \mathbf{s} = \widehat{\mathbf{s}}, \\ F_j\big(\mathbf{s}, \mathbf{g}_j^{n-1}(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big) & \text{otherwise.} \end{cases}$$

By induction on $n \geq 0$, we will show that the following holds for all $n \geq 0$.

$$g_j^n(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) \in \big[d_j(\mathbf{s}), d_j(\mathbf{s}) + \mu\big] \text{ for all } \mathbf{s} \in \mathscr{S}. \tag{34}$$

Let $\mathbf{g}_j^0(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) = \mathbf{d}_j$. Since $\mu \geq 0$, this implies that (34) holds for $n = 0$. Now, suppose (34) holds for some $n = m \geq 0$. Now, for an arbitrary $\mathbf{s} \in \mathscr{S}$ consider $g_j^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) - d_j(\mathbf{s})$.

1. If $\mathbf{s} \neq \widehat{\mathbf{s}}$, then $g_j^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) - d_j(\mathbf{s}) = \lambda_j \sum_{\mathbf{s}' \in \mathscr{S}} \mathscr{P}(\mathbf{s}'|\mathbf{s})\big[g_j^m(\mathbf{s}', \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) - d_j(\mathbf{s}')\big]$. Since $\sum_{\mathbf{s}' \in \mathscr{S}} \mathscr{P}(\mathbf{s}'|\mathbf{s}) = 1$, $\lambda_j < 1$ and $\mu \geq 0$, by the induction hypothesis, this implies $\big[g_j^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) - d_j(\mathbf{s})\big] \in \big[0, \mu\big]$.

2. If $\mathbf{s} = \widehat{\mathbf{s}}$, then $g_j^{m+1}(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) - d_j(\mathbf{s}) = u_j(\mathbf{s}, 1) - d_j(\mathbf{s}) = \mu \geq 0$.

Thus, (34) holds for $n = m+1$. Then, by the convergence of value iteration, (33) holds. Next, we will show that $u_j(\widehat{\mathbf{s}},1) \geq F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big)$.

$$\begin{aligned} u_j(\widehat{\mathbf{s}},1) - F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big) &= u_j(\widehat{\mathbf{s}},1) - F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big) - d_j(\widehat{\mathbf{s}}) + F_j(\widehat{\mathbf{s}}, \mathbf{d}_j) \\ &= \mu - \lambda_j \sum_{\mathbf{s}' \in \mathscr{S}} \mathscr{P}(\mathbf{s}'|\widehat{\mathbf{s}})\big[g_j(\mathbf{s}', \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) - d_j(\mathbf{s}')\big] \geq \mu - \lambda_j \sum_{\mathbf{s}' \in \mathscr{S}} \mathscr{P}(\mathbf{s}'|\widehat{\mathbf{s}})\mu = (1-\lambda_j)\mu \geq 0, \end{aligned}$$

where the first inequality follows from the fact that $g_j(\mathbf{s}', \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) \leq d_j(\mathbf{s}') + \mu$ for all $\mathbf{s}' \in \mathscr{S}$ $\big($by (33)$\big)$ and the second inequality is implied by the facts that $\lambda_j < 1$ and $\mu \geq 0$.

Thus, $u_j(\widehat{\mathbf{s}},1) \geq F_j\big(\widehat{\mathbf{s}}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big)$. Then, by (32), the payoffs $\mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})$ can be restated as:

$$g_j(\mathbf{s}, \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j}) = \begin{cases} \max\big\{u_j(\mathbf{s},1), F_j\big(\mathbf{s}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big)\big\} & \text{if } \mathbf{s} = \widehat{\mathbf{s}}, \\ F_j\big(\mathbf{s}, \mathbf{g}_j(\widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_{-j})\big) & \text{otherwise.} \end{cases}$$

By the construction of the strategy profile $\widehat{\mathbf{A}}$, for Patient j, this implies that recursion (3) is satisfied for all $\mathbf{s} \in \mathscr{S}$. Now, we will establish the main result. Consider the two possible cases for $\max_{i \in \mathcal{N}} \big[d_i(\widehat{\mathbf{s}}) - u_i(\widehat{\mathbf{s}},1)\big]$.

1. If $\max_{i \in \mathcal{N}} \big[d_i(\widehat{\mathbf{s}}) - u_i(\widehat{\mathbf{s}},1)\big] > 0$, then by (31), $\max_{i \in \mathcal{N}} \big[g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) - u_i(\widehat{\mathbf{s}},1)\big] > 0$.

2. If $\max_{i \in \mathcal{N}} \big[d_i(\widehat{\mathbf{s}}) - u_i(\widehat{\mathbf{s}},1)\big] \leq 0$, then since $\Pi_{\widehat{\mathbf{s}}} \neq \emptyset$, $\sum_{i \in \mathcal{N}} g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) \geq \sum_{i \in \mathcal{N}} u_i(\widehat{\mathbf{s}},1)$. Now, consider the following subcases:

   (a) If $\sum_{i \in \mathcal{N}} g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) > \sum_{i \in \mathcal{N}} u_i(\widehat{\mathbf{s}},1)$, then $\max_{i \in \mathcal{N}} \big[g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) - u_i(\widehat{\mathbf{s}},1)\big] > 0$.

   (b) If $\sum_{i \in \mathcal{N}} g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) = \sum_{i \in \mathcal{N}} u_i(\widehat{\mathbf{s}},1)$, then either $g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) = u_i(\widehat{\mathbf{s}},1)$ for all $i \in \mathcal{N}$ or $g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) \neq u_i(\widehat{\mathbf{s}},1)$ for some $i \in \mathcal{N}$. Because $\sum_{i \in \mathcal{N}} g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) = \sum_{i \in \mathcal{N}} u_i(\widehat{\mathbf{s}},1)$, in the latter case there must exist some $j \in \mathcal{N}$ with $g_j(\widehat{\mathbf{s}}, \mathbf{a}_j^*, \mathbf{a}_{-j}^*) > u_j(\widehat{\mathbf{s}},1)$ so that $\max_{i \in \mathcal{N}} \big[g_i(\widehat{\mathbf{s}}, \mathbf{a}_i^*, \mathbf{a}_{-i}^*) - u_i(\widehat{\mathbf{s}},1)\big] > 0$.

□

Appendix B: Numerical Results on Three-Way Exchanges

In this appendix, we present numerical results on three-way exchanges that are similar in nature to our study of two-way exchanges in Section 4. We aggregate the two-way GFR ranges 1, 2, 3 and 4 into a single range, and GFR ranges 5, 6 and 7 into another range, so that $\Phi = \{1, 2, 3, 4\}$ in our experiments for three-way exchanges. We let $\widehat{\mathbf{s}} = (\widehat{s}_1, \widehat{s}_2, \widehat{s}_3)$ refer to the initial state of the game, *i.e.*, the state in which patients are matched. For each of the 500 cases we solve the mathematical programs $\max_{(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}})\in\Lambda} \left[\sum_{i\in\mathcal{N}} w_i(\widehat{\mathbf{s}})\right]$ and $\max_{(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}})\in\Upsilon} \left[\sum_{i\in\mathcal{N}} w_i(\widehat{\mathbf{s}})\right]$ to estimate the cost of restricting our attention to pure strategies, for which $\max_{(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}})\in\Lambda} \left[\sum_{i\in\mathcal{N}} w_i(\widehat{\mathbf{s}})\right] - \max_{(\widehat{\mathbf{w}},\widehat{\mathbf{y}},\widehat{\mathbf{z}})\in\Upsilon} \left[\sum_{i\in\mathcal{N}} w_i(\widehat{\mathbf{s}})\right]$ provides an upper bound (by Theorems 2 and 3). Similar to the two-way case, our experiments show that a socially optimal randomized equilibrium does not a provide a significant welfare over a socially optimal pure equilibrium. In all our instances, the socially optimal pure equilibrium was never more than 1.12% worse than the socially optimal randomized equilibrium. Therefore, we quantify the welfare loss due to patient autonomy only for socially optimal pure equilibria. We define the maximum difference in patients' GFR ranges in the initial state of the game $\left(\max(\widehat{s}_1, \widehat{s}_2, \widehat{s}_3) - \min(\widehat{s}_1, \widehat{s}_2, \widehat{s}_3)\right)$ as a measure of conflict among patients' self interests. Therefore, we present only the healthiest and sickest patients where Patient i is sickest if $\widehat{s}_i \geq \widehat{s}_j$ for both $j \in \mathcal{N}_{-i}$, and healthiest if $\widehat{s}_i \leq \widehat{s}_j$ for both $j \in \mathcal{N}_{-i}$. Note that these definitions of sickest or healthiest patients may not be unique. For each case, we calculate the social welfare loss and the individual patients' welfare losses due to patient autonomy in absolute (in quality-adjusted life days) and relative terms as a function of the maximum difference in their GFR ranges. When there are multiple sickest and healthiest patients, we consider the average welfare terms across such patients.

From Table 3, it easy to see that the pattern in welfare loss borne by patient autonomy in the difference of patients' GFR ranges is similar to that in two-way kidney exchanges. As patients' health statuses diverge, the sickest patient loses more whereas the healthiest patient loses less, and eventually gains from patient autonomy. Notice that three-way exchanges incur less social welfare loss from patient autonomy than two-way exchanges, which we interpret as follows: In three-way exchanges, the death

Table 3: Social welfare loss and patients' individual welfare losses due to patient autonomy.

	Relative Loss				Absolute Loss			
$\max(\widehat{s}_1, \widehat{s}_2, \widehat{s}_3) - \min(\widehat{s}_1, \widehat{s}_2, \widehat{s}_3)$	0	1	2	3	0	1	2	3
Healthiest Patient	0.76	0.29	0.13	-1.32	46	17	11	-65
Sickest Patient	0.76	0.84	1.32	7.60	46	52	65	366
Total	0.40	0.56	0.72	2.55	68	93	117	400

of any of the three patients renders an exchange infeasible. Therefore, the healthier patient(s) is (are) less patient than in the case of two-way exchanges and force(s) less the other(s) to wait. By the same reasoning, because the socially optimal policy is more eager to exchange, the welfare loss from not exchanging the kidneys immediately due to patient autonomy is less than the loss depicted in Table 3 and the analogous loss incurred in two-way exchanges. Lastly, similar to two-way exchanges, we also quantify the patients' cost of playing a socially optimal equilibrium rather than their individually optimal equilibria. In all our instances, under a socially optimal equilibrium none of the patients was more than 1.15% worse relative to their individually optimal equilibria.

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