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Vulnerability Measurement of Supply Chain Networks from Topology Perspective

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Abstract: This paper studies vulnerability measurements for supply chain networks. A continuous model for supply chain is established and metrics that can quantify the vulnerability are proposed from topology perspective. Suggestions to increase vigilance against risks of critical nodes and paths in the network are stated based on these metrics.

Keywords: Supply chain vulnerability; Complex network; Continuous modeling

1. Introduction

In today's shifting business environment, modern supply chains have become more vulnerable than ever before. Two reasons account for this. First, supply chain vulnerability is related to its structure and management level. With the development of globally expanded supply chain network, manufacturing outsourcing, increased dependence on supplier capacities, more required regulatory, supply chain management has become more complex. These facts increase supply chain risks. Facing these problems, supply chain executives have implemented numerous strategies such as reduced supply base, just-in-time inventory system, or OEM with the purpose to reduce costs and improve profits [1-4]. These activities have been proved efficient in a stable environment but do make supply chains more exposed to risks, as supply chain has been left with fewer buffers. Hence even a slight disturbance may cause supply chain disruption. Second, disasters, both natural and man-made, have increased in frequency and intensity during the past decades. Natural disasters such as earthquakes, hurricanes, droughts, floods, windstorms and tsunamis strike more often and have greater impact on social economics [5]. Meanwhile, man-made disasters such as accidents, wars, terrorist attacks, or strikes have threatened supply chain development [6]. For instance, the 2002 West Coast port lockout caused a shutdown of the port for six months and this in turn raised at least $24 million loss for the Cosco Group, China.

A growing number of academicians and practitioners have therefore paid attention to supply chain risk and its vulnerability. Concepts of supply chain vulnerability have been proposed in recent publications [7-9]. Christopher and Peck defined supply chain vulnerability as "an exposure to serious disturbance" [10]. Sevensson distinguished between atomistic vulnerability (of a part of the supply chain) and holistic vulnerability (across the entire supply chain) [9]. Wagner and Bode postulated that supply chain vulnerability is a function of certain supply chain characteristics and that the loss a firm has is a result of its supply chain vulnerability to a given supply chain disruption [11]. In a later study, they defined the concept of supply chain vulnerability more precisely: supply chain vulnerability is composed of two aspects. One is supply chain characteristics, contributing to its susceptibility to harms. The other is the impact of supply chain disruptions, including the probability of occurrence and the severity [12].

Despite the reasons and definition of supply chain vulnerability have been explored, ways of measuring the inherent vulnerability of supply chain remains a pressing challenge. As mentioned in [13], supply chain executives need methods to measure and quantify supply chain vulnerability in order to manage those properly. The objective of this research is therefore to introduce a quantitative approach using continuous modeling to measure supply chain vulnerability. To achieve this, a continuous model is established based on the topology structure of supply chain network.

The remainder of this paper is organized as follows. In Section 2 the continuous model that models the flow of products in the supply chain network is established. In Section 3, measurements for supply chain vulnerability are proposed and formulated based on the continuous model. An example is given to illustrate the application of the measurements in Section 4. The results of some numerical examples are discussed in Section 5. The paper concludes in Section 6 with an outlook and directions of future research.

2. The model

In order to quantify the vulnerability by developing and calculating a supply chain vulnerability measurement, we establish a model for the supply chain network. Continuous models for supply chains have been discussed largely in recent literatures [14-22]. Those models are based on fluid-dynamic considerations and they rely on the fact that in many applications only averaged quantities need to be modeled. We refer to the cited literatures and references therein for an overview and relations to classical queuing and discrete event models [23-25]. We start the investigation based on a recent, general continuous supply chain model introduced in [20]. The setup is as follows. The supply chain network is modeled as a directed graph (ν, ε), where $i \in \nu$ are the nodes, i.e., suppliers, and $e = (i, j) \in \varepsilon$ a link, i.e., a supply-demand relationship. The edge $e = (i, j)$ connects supplier i and j. Depending on the edge, we distinguish between the edges $e \in \varepsilon_{out} \subset \varepsilon$ where products leave the system, $e \in \varepsilon_{in} \subset \varepsilon$ where products enter the system and all other links. Each node represents a supplier. Each supplier is characterized by a maximal production capacity $\mu_i > 0$ and a production rate $1/\tau_i > 0$. The number of products at time t within node i is denoted by $u_i(t)$. The basic relation between the number of produced products per second (flow) $f_i(t)$ and the number of products $u_i(t)$ within the node is governed by a so-called fundamental diagram. Several different fundamental diagrams have been proposed in the literature and its particular shape depends on the context. Here, we use the following relation corresponding to a M/M/1 queue:

$$f_i(t) = \min\{\mu_i, u_i(t)/\tau_i\}. \tag{1}$$

We refer to [14, 19, 26] for a detailed derivation. We denote by $\vec{u}$ the vector $\vec{u} = (u_i)_{i \in \nu}$ and similarly for $\vec{f}$. The produced parts of supplier *i* are shipped to a connected supplier *j*. The incidence matrix of the graph is denoted by *A*. Its coefficient a_{ij} describes the distribution rate of products from supplier *j* to *i*. Then, the full continuous model of the supply network is given by Equation (1), (2) and (3).

$$\frac{d\vec{u}(t)}{dt} = A\vec{f}(t) - \vec{f}(t) + \vec{f}_{ext}(t), \qquad \vec{u}(0) = 0 \tag{2}$$

Here, we assume for simplicity that the network initial empty. The connectivity matrix $A \in R_+^{|\nu|\times|\nu|}$ has entries $A = (a_{ij}) \in [0,1]$. It is reasonable to assume that the value a_{ij} is given by production capacities of suppliers, who receive products from supplier j , i.e., we have

$$a_{ij} = \frac{w_{ji}\mu_j}{\sum_i w_{ji}\mu_j} \tag{3}$$

Here, $w_{ji} = 1$, if there exists the link (j,i) in the graph of the network and $w_{ji} = 0$ otherwise.

At some nodes i we have a given inflow profile $\vec{f}_{ext}(t)$. The Equation (2) therefore describes the rate of change in products over time in supplier i depending on the incoming products $A\vec{f}$, the outgoing products $-\vec{f}$ and possible external inflows $\vec{f}_{ext}$. The continuous model conserves all products within the network due to Equation (2).

Remark 1.*Equation (2) can be easily extended to include delays* $1/v_e$ *due to transportation on link* $e = (i, j)$ *. In this case, we consider instead of (2) equation*

$$\frac{d\vec{u}(t)}{dt} = A\vec{f}(t - \frac{1}{v_e}) - \vec{f}(t) + \vec{f}_{ext}(t) . \tag{4}$$

The later can also be seen as a particular discretization of a conservation law governing the flow on link e *. We refer to [18] for more details.*

3. Vulnerability measures

Supply chain vulnerability can be measured and managed at different levels, e.g. on the scale of an entire economy, an industry, a supply chain, or only at a local company [13]. Managers at different levels should define different measures and take different strategies to reduce supply chain vulnerability. In this paper, we would like to focus on the level of a supply chain network.

Based on the basic supply chain model (1, 2, 3), we now introduce vulnerability measures for the entire supply chain network. The products arriving at customers on a time horizon $[0,T]$ is given by

$$J(T,\upsilon)=\int_0^T \sum_{(i,j)\in\varepsilon_{out}} f_j(t)dt \tag{5}$$

Clearly, as soon as the external inflow $\vec{f}_{ext}(t)$ is known, the value of Formulation (5) is known. The value depends in particular on the distribution matrix A and therefore on the production capacities μ_j. In the presented supply chain model, a disturbance to node j means the capacity μ_j is decreased to a new capacity $\mu_j^{'}\in[0,\mu_j)$.The influence of the capacities μ_j on J is measured by analyzing the sensitivity of J with respect to μ_j, hence, we need to compute $\delta_j J$

$$\delta_j J := \frac{\partial J}{\partial \mu_j}(T,\upsilon) \tag{6}$$

The concept of sensitivity varies across authors [27]. We use its general sense, the degree to which the system is modified or affected by an internal or external disturbance or set of disturbances. For each node $j\in\upsilon$, the value $\delta_j J$ specifies whether or not the number of products arriving at customers is sensitive to the change of production capacities of each node. This allows to determine vulnerable nodes (i.e. suppliers) and is used to measure the supply chain vulnerability influenced by its nodes.

Several other vulnerability measures can be introduced base on $\delta_j J$. First would be to look for vulnerable routings inside the network. We can find paths which connecting the entrance nodes and the final destination nodes. The vulnerability of these paths can be computed e.g. by summing the sensitivities of the single nodes and thereby allow to identify possible weak links in the network.

Second possible measure would be to use the L_∞ norm of all the sensitivity values. We look for a single node which is most vulnerable in the supply chain network through this method. Networks with a low L_∞ norm value are robust against attacks on any single node.

Another possible measure would be to use the L_2 norm of all the sensitivity values. A low L_2 norm indicates that these networks are robust against failure of many nodes. Hence, those are quite stable if there are multiple failures.

4. Illustrative example

In this section, a simple network example is used to discuss the applicability of the quantitative approach to vulnerability. Fig.1 depicts a sample supply chain network. Node S_1 is the raw material supplier, and it has an external inflow f_{ext} equaling to 40. Node S_6 represents the customers. Nodes S_2 through S_5 are the suppliers which can be for example, manufacturers, distributors, or retailers, that serve as connection nodes for the supply chain network. The arrows point to the direction in which products flow from raw material suppler to customers. The maximal production capacity of node S_1 to node S_6 is given by $[40,30,10,20,20,40]$. The production rate of each node is assumed to be the same as 1 and time horizon $[0,T]$ supposed to be $[0,1]$.

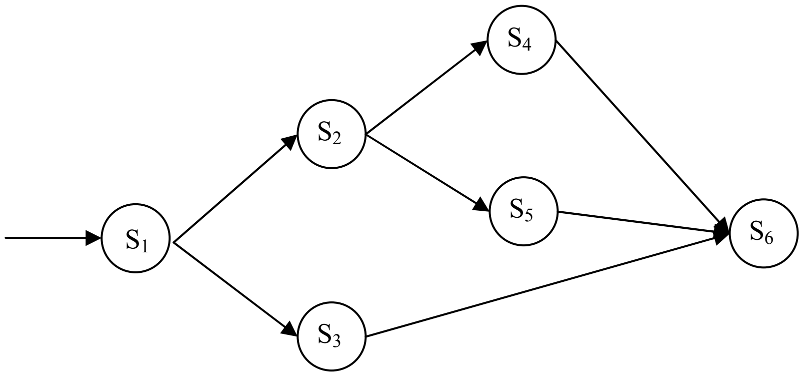

Fig.1 Example of a supply chain network

In normal situation, the supply chain network usually works at a steady-state, which means the number of products in each supplier in the network keeps constant, i.e. $d\vec{u(t)}/dt = 0$. In this case, we have from Equation (2)

$$f = (I_d - A)^{-1} f_{ext} T\ . \tag{7}$$

and Equation (5) simplifies to

$$J(T,\upsilon) = \sum_{(i,j)\in\varepsilon_{out}} ((I_d - A)^{-1} f_{ext})_j T\ . \tag{8}$$

According to Equation (3), distribution matrix A is calculated as follows.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu_2}{\mu_2+\mu_3} & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu_3}{\mu_2+\mu_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_4}{\mu_4+\mu_5} & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_5}{\mu_4+\mu_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{9}$$

Hence, number of products arriving at customers during $[0,T]$ is equal to

$$J(T,\upsilon) = \sum_{(i,j)\in\varepsilon_{out}} ((I_d - A)^{-1} f_{ext})_j T = f_6 = (a_{63}a_{31} + a_{64}a_{42}a_{21} + a_{65}a_{52}a_{21}) f_{ext} T$$

$$= \left(\frac{\mu_3}{\mu_2+\mu_3} + \frac{\mu_4}{\mu_4+\mu_5}\frac{\mu_2}{\mu_2+\mu_3} + \frac{\mu_5}{\mu_4+\mu_5}\frac{\mu_2}{\mu_2+\mu_3}\right) f_{ext} T \qquad (10)$$

Disturbance to any node in the supply chain network will cause a rerouting of products among the residual nodes. Since each node has a maximal production capacity, it may not sufficient to process the rerouted products flowing into it. In this case, the outflow of the entire supply chain network may be affected, and there might be not enough products to meet customers' demands. We use the measurement $\delta_j J$ to explain how the node disturbances affect the outflow of the entire supply chain network.

Since products rerouting in the network may exceed the capacities of some nodes, we modify Equation (10) according to Equation (1), and get Equation (11) as follows.

$$J(T,\upsilon) = f_6 T = (f_3 + f_4 + f_5)T = (\min\{\mu_3, a_{31} f_1\} + \min\{\mu_4, a_{42} f_2\} + \min\{\mu_5, a_{52} f_2\})T$$

$$= (\min\{\mu_3, a_{31} f_1\} + \min\{\mu_4, a_{42} \min\{\mu_2, a_{21} f_1\}\} + \min\{\mu_5, a_{52} \min\{\mu_2, a_{21} f_1\}_2\})T$$

$$= (\min\{\mu_3, a_{31} \min\{\mu_1, f_{ext}\}\} + \min\{\mu_4, a_{42} \min\{\mu_2, a_{21} \min\{\mu_1, f_{ext}\}\}\}$$

$$+ \min\{\mu_5, a_{52} \min\{\mu_2, a_{21} \min\{\mu_1, f_{ext}\}\}\})T \qquad (11)$$

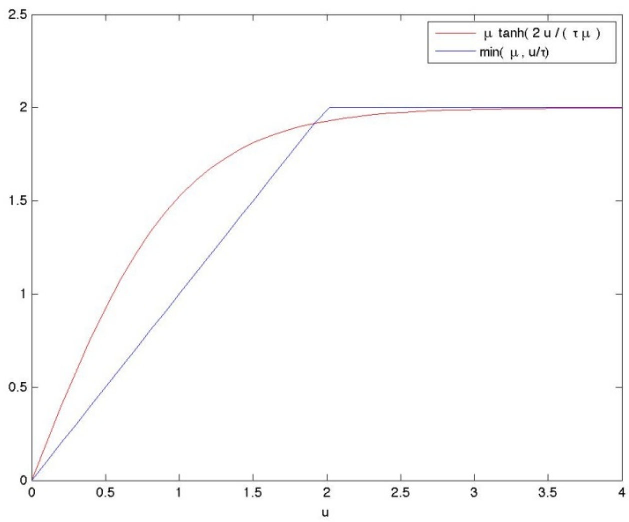

Fig.2 approximation curves

Here, Equation (11) is a multi-layer nested piecewise function and it's not differentiable. However, we may approximate the function $\min(\mu, u/\tau)$ by $\mu\tanh(2u/\tau\mu)$. This is a smooth approximation as seen in Fig.2 for the example of $\mu = 2$ and $\tau = 1$.

Hence, Equation (11) can be approximated as follows.

$$J(T,\upsilon) = \mu_3 \tanh\left(\frac{2a_{31}\mu_1 \tanh\left(\frac{2f_{ext}}{\mu_1}\right)}{\mu_3}\right)T + \mu_4 \tanh\left(\frac{2a_{42}\mu_2 \tanh\left(\frac{2a_{21}\mu_1 \tanh(\frac{2f_{ext}}{\mu_1})}{\mu_2}\right)}{\mu_4}\right)T$$

$$+ \mu_5 \tanh\left(\frac{2a_{52}\mu_2 \tanh\left(\frac{2a_{21}\mu_1 \tanh(\frac{2f_{ext}}{\mu_1})}{\mu_2}\right)}{\mu_5}\right)T \quad (12)$$

The derivatives $\delta_j J, \forall j$ can be calculated explicitly and are given in Appendix A. The value of $\delta_j J$ shows how does disturbance of node j affect the whole supply chain network. Table 1 shows the vulnerability values for each node when it is disturbed and products are rerouted over the residual capacities of remaining nodes.

Table 1 Sensitivities of J with respect to the nodes

nodes	vulnerability values
S_1	0.0738
S_2	0.3009

S_3	0.8721
S_4	0.6027
S_5	0.6027

In our simple example the material can travel either along the nodes 1-3-6, 1-2-4-6 or 1-2-5-6. We simply compute the sum along the paths we observe that the path 1-2-4-6 and 1-2-5-6 are more vulnerable than 1-3-6. This is clearly due to the fact that here more nodes are present increasing the risk.

In the case of L_∞ norm, the vulnerability of the previous supply chain is then given by $\max(\delta_j J) = 0.8721$ and the most vulnerable node is S_3.

In the case of L_2 norm, we have $\sqrt{\sum_{j=1}^{5} \delta_j J}$ =1.2582 as possible measure of the system. A low value would indicate a robust supply network.

5. Discussion and implications

In this article we proposed a quantitative method for supply chain vulnerability measurement. Supply chain managers may find the vulnerable nodes and paths that have significant influence on the overall outflow of the entire supply chain network through this method. They can set priorities and execute appropriate measures to mitigate possible disturbances at these nodes and paths. Since it costs much to manage risks of the whole supply chain, the managers should allocate more management resources at the most vulnerable nodes and paths. The method presented in this paper can help supply chain managers recognize vulnerable nodes and paths in the network and make effective and efficient decisions about risk management resources allocation.

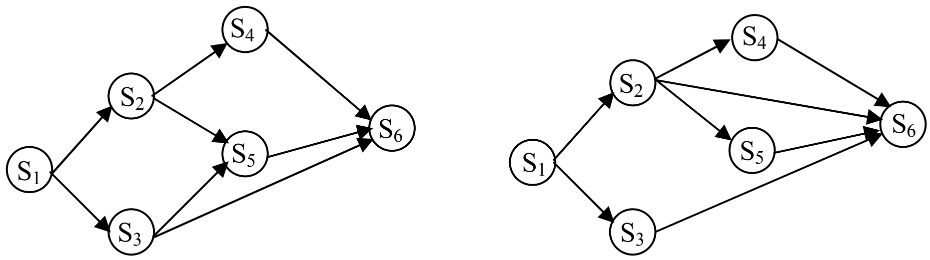

Fig.3 Two other examples of supply chain network

Table 2 comparison of vulnerability of supply chain networks with different topologies

	μ_1	μ_2	μ_3	μ_4	μ_5	Most vulnerable paths	L_∞ norm	L_2 norm
1#	0.0738	0.3009	0.8721	0.6027	0.6027	1-2-4-6 1-2-5-6	0.8721	1.2582
2#	0.0562	0.2354	0.6449	0.6673	0.5978	1-3-5-6	0.6673	1.1301
3#	0.1455	0.9033	0.788	0.2136	0.2136	1-2-4-6 1-2-5-6	0.9033	1.2447

In the illustrative example we calculate vulnerability for each node, and find the most vulnerable nodes and paths. We do some experiments on two other supply chain networks, which are shown in Fig.3. All the three supply chain networks have the same node set v, but a slightly different link set ε. We compute vulnerability for nodes in each of the network and make a comparison in Table 2. We can observe that even though there is a slightly difference between the network topologies, the results are obviously different. Consequently, the vulnerability measures are greatly depending on the topology of the supply chain network. They distinguish the vulnerability among different network topologies very well.

We further do some experiments using different parameters as stated in Table 3. In each experiment, we choose a node and gradually enlarge its capacity, while fixing capacities of other nodes. Then we calculate vulnerability values of nodes for each experiment. The results are shown in Fig.4 to Fig.8. From Fig.4 to Fig.7 we observe that increasing the capacities of nodes decrease its own vulnerability, but may have a promotion or restraining effect on other nodes. Hence, increasing the capacity of a node decreases its own vulnerability but have an un-determinate effect on the vulnerability of the entire supply chain network, as shown in Fig.8. Here we take L_2 norm measure as an example.

Table 3 parameters of different experiments

	μ_1	μ_2	μ_3	μ_4	μ_5
Ex.1	10, 20, 30, 40, 50	30	10	20	20
Ex.2	20	10, 20, 30, 40, 50	20	20	20
Ex.3	40	30	10, 20,30, 40, 50	20	20
Ex.4	40	30	10	10, 20,30, 40, 50	20

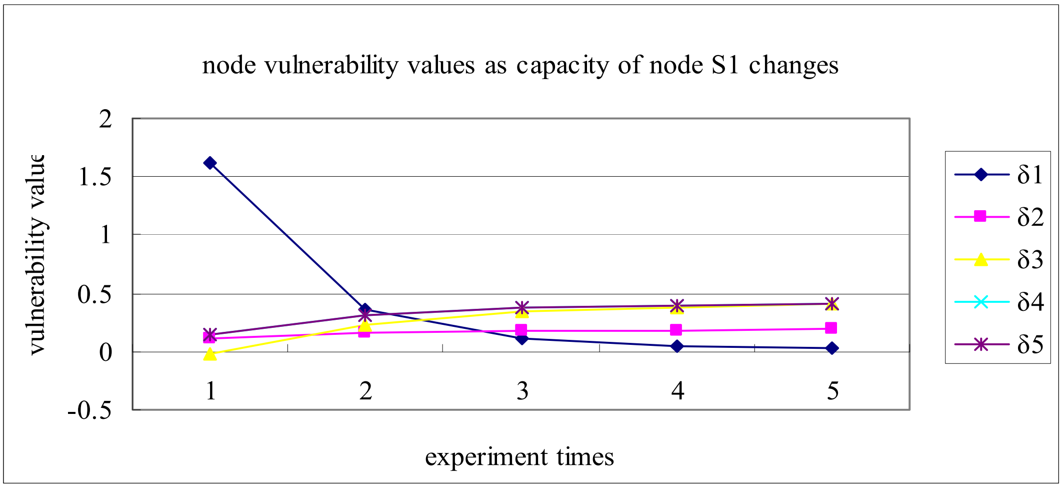

Fig.4 Node vulnerability values of all the nodes as μ_1 changes

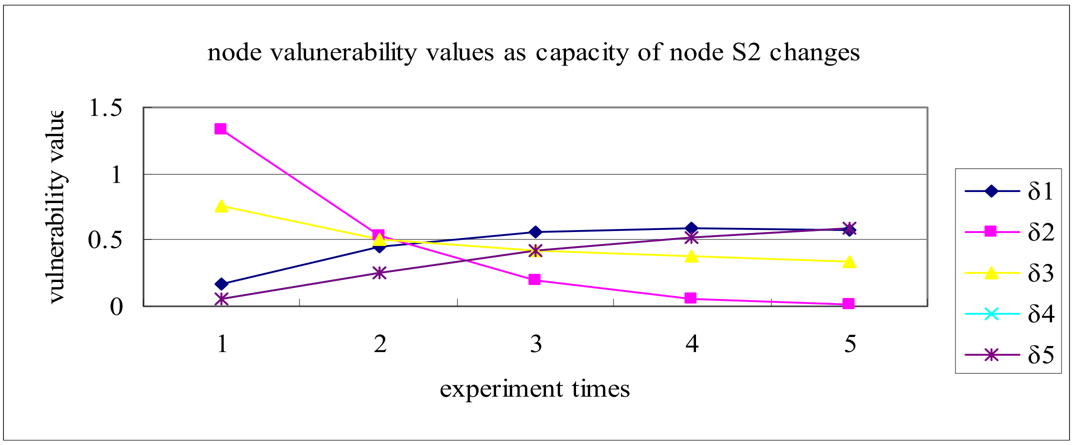

Fig.5 Node vulnerability values of all the nodes as μ_2 changes

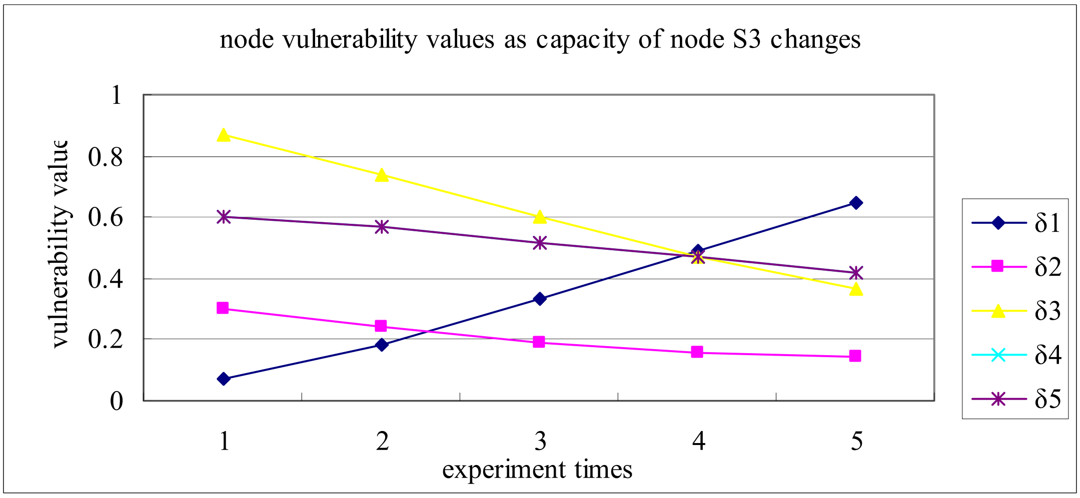

Fig.6 Node vulnerability values of all the nodes as μ_3 changes

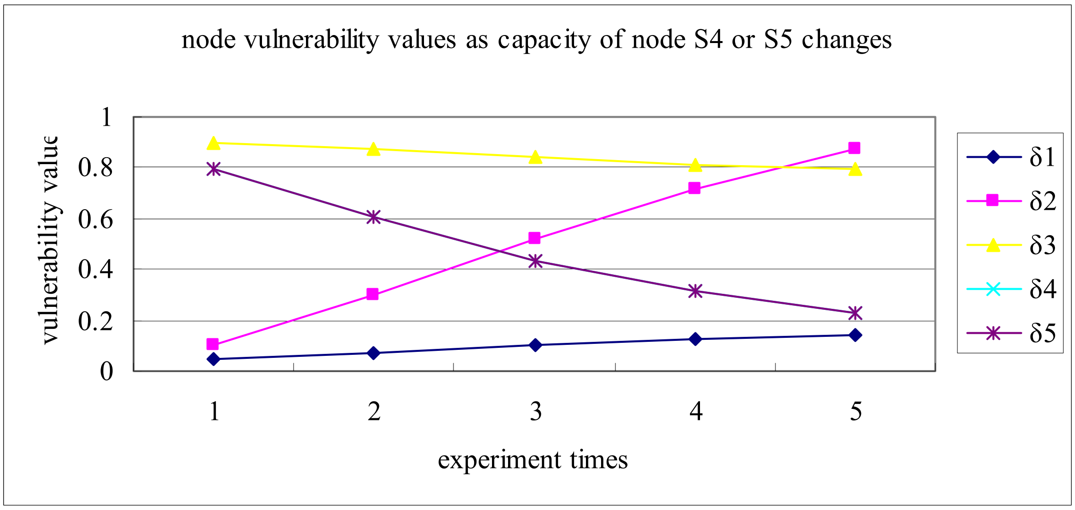

Fig.7 Node vulnerability values of all the nodes as μ_4 or μ_5 changes

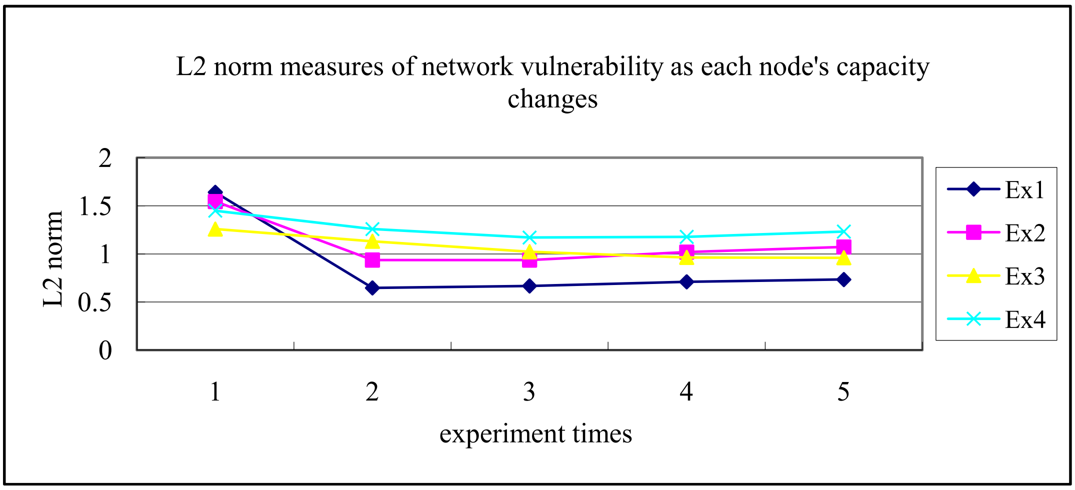

Fig.8 L_2 norm measures as each node's capacity changes

6. Conclusions

In this paper we presented a quantitative method to measure vulnerability of supply chain network. A supply chain network model was established by utilizing the continuous modeling and the vulnerability measurement was deduced from the model. Some examples were also given to illustrate the usage of the method. This research did an attempt to quantify the supply chain vulnerability and gave some management suggestions based on this measurement. So far, only the steady-state case has been analyzed.

Despite this measurement, our research is not without limitations. First, only the topology of the supply chain network is considered in this research. It has been simplified to a network with products flowing through it. In the future research, other factors of the supply chain such as the management activities should be included. Second, this research has provided a method to measure the vulnerability of an already formative supply chain network. It may be extended to give guidance to supply chain network design. Third, as discussed in Section 5, how to decrease vulnerability of the entire supply chain network through decreasing vulnerability of each node in the network remains a problem to be solved in future researches.

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Appendix A: Expressions of measures $\delta_j J$ of example in section 4

$$\delta_1 J = \frac{\partial J}{\partial \mu_1}$$

$$= \left(2\mu_2\mu_4 \left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)}{\mu_2 + \mu_3} \right)}{\mu_4 + \mu_5} \right)^2 - 1 \right) \left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)}{\mu_2 + \mu_3} \right)^2 - 1 \right) \left(\frac{2 \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)}{\mu_2 + \mu_3} + \frac{4 f_{ext} \left(\tanh\left(\frac{2 f_{ext}}{\mu_1} \right)^2 - 1 \right)}{\mu_1 (\mu_2 + \mu_3)} \right) \right) / (\mu_4 + \mu_5)$$

$$- \mu_3 \left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)}{\mu_2 + \mu_3} \right)^2 - 1 \right) \left(\frac{2 \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)}{\mu_2 + \mu_3} + \frac{4 f_{ext} \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)^2 - 1}{\mu_1 (\mu_2 + \mu_3)} \right)$$

$$+ \left(2\mu_2\mu_5 \left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)}{\mu_2 + \mu_3} \right)}{\mu_4 + \mu_5} \right)^2 - 1 \right) \left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)}{\mu_2 + \mu_3} \right)^2 - 1 \right) \left(\frac{2 \tanh\left(\frac{2 f_{ext}}{\mu_1} \right)}{\mu_2 + \mu_3} + \frac{4 f_{ext} \left(\tanh\left(\frac{2 f_{ext}}{\mu_1} \right)^2 - 1 \right)}{\mu_1 (\mu_2 + \mu_3)} \right) \right) / (\mu_4 + \mu_5)$$

$$\delta_2 J = \frac{\partial J}{\partial \mu_2}$$

$$= \frac{2\mu_1\mu_3 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)\left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)^2 - 1\right)}{(\mu_2+\mu_3)^2} - \mu_5\left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)^2 - 1\right)$$

$$\left(\frac{2\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5} + \frac{4\mu_1\mu_2 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)\left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)^2 - 1\right)}{(\mu_2+\mu_3)^2(\mu_4+\mu_5)}\right)$$

$$- \mu_4\left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)^2 - 1\right)$$

$$\left(\frac{2\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5} + \frac{4\mu_1\mu_2 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)\left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)^2 - 1\right)}{(\mu_2+\mu_3)^2(\mu_4+\mu_5)} \right)$$

$$\delta_3 J = \frac{\partial J}{\partial \mu_3}$$

$$= \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right) + \frac{2\mu_1\mu_3 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)\left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)^2 - 1\right)}{(\mu_2+\mu_3)^2}$$

$$- \left(4\mu_1\mu_2\mu_4 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)\left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)^2 - 1\right) \left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)^2 - 1\right) \right) / (\mu_2+\mu_3)^2(\mu_4+\mu_5)$$

$$
-\left(4\mu_1\mu_2\mu_5 \tanh\left(\frac{2f_{ext}}{\mu_1}\right)\left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)^2 -1\right)\left(\tanh\left(\frac{2\mu_1 \tanh\left(\frac{2f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)^2 -1\right)\right)/(\mu_2+\mu_3)^2(\mu_4+\mu_5)
$$

$$
\delta_4 J = \frac{\partial J}{\partial \mu_4}
$$

$$
= \tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)
$$

$$
+\frac{2\mu_2\mu_4 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)\left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)^2 -1\right)}{(\mu_4+\mu_5)^2}
$$

$$+\frac{2\mu_2\mu_5 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)\left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)^2-1\right)}{(\mu_4+\mu_5)^2}$$

$$\delta_5 J = \frac{\partial J}{\partial \mu_5}$$

$$= \tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)$$

$$+\frac{2\mu_2\mu_4 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)\left(\tanh\left(\frac{2\mu_2 \tanh\left(\frac{2\mu_1 \tanh\left(\frac{2 f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)^2-1\right)}{(\mu_4+\mu_5)^2}$$

$$+\frac{2\mu_2\mu_5\tanh\left(\frac{2\mu_1\tanh\left(\frac{2f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)\left(\tanh\left(\frac{2\mu_2\tanh\left(\frac{2\mu_1\tanh\left(\frac{2f_{ext}}{\mu_1}\right)}{\mu_2+\mu_3}\right)}{\mu_4+\mu_5}\right)^2-1\right)}{(\mu_4+\mu_5)^2}$$