Abstract Number: 025-0620

A New Approach to Design of Fuzzy Multi-Attribute Control Charts

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POMS 23rd Annual Conference

Chicago, Illinois, U. S. A.

April 20 to April 23, 2012.

Abstract: This paper designs control charts for subjectively estimated multiple-attribute quality characteristics. Aggregate sample quality is estimated by using interactive weighted addition of fuzzy values assigned to each quality characteristic. Control charts are drawn using possibility and necessity measures. This approach helps to identify quality characteristics responsible for out-of-control situations.

Key words: Multi-attribute and correlated quality characteristics, Interactive weighted addition of fuzzy values, Possibility and necessity measures, and Fuzzy control charts

1. Introduction

Statistical process control concepts and methods have been very important in the manufacturing and process industries. Their principal objective is to monitor the performance of a process over time in order for the process to achieve a state of statistical control. Such a state of control is said to exist if the quality characteristics remain close to their desired values and the only source of their variation is the common cause. Shewhart and CUSUM, both univariate and multivariate control charts, are commonly used to monitor key quality characteristics in order to detect the occurrence of any event having a "special" or assignable cause. The power of control charts lies in their ability to detect process shifts and process variability, thus identifying abnormal conditions in a production process.

Often the quality characteristics of interest could be correlated and many in number, and they could need subjective assessment by quality experts. Researchers in the past have considered these features separately. In this paper, we integrated these concepts and suggest an approach for designing control charts for multiple- attribute type quality characteristics whose values are estimated subjectively. The approach also provides a method to identify the specific variables that cause a process to go out of control.

The paper is organized as follows: The next section makes a review of literature on multivariate and fuzzy multivariate control charts. Section 3 gives the principles underlying the detection of out-of-situation process conditions. Section 4 presents two examples to illustrate the application of the model to detect in-control and out-of-control conditions of a process and to draw fuzzy control charts. Section 5 discusses the implications of the model and indicates future scope of work.

2. Literature Review

Shewhart control charts provide a good method for identifying the existence of extra variability due to assignable causes and for detecting problems and changes in manufacturing processes that may not be detected otherwise. Quite often, the quality of a product depends on more than one quality attribute that are, in general, correlated. In such cases, a multivariate control procedure that considers the correlation structure between the attributes is appropriate.

Many forms of multivariate control charts are available in the literature, three notable multivariate control charts being Hotelling's T^2 charts, multivariate cumulative sum control charts, and multivariate exponential weighted moving average charts. Early research on multivariate shewhart charts goes back to Hotelling **[6]**, who introduced the problem of a correlation between the quality characteristics of a process and came up with the T^2 statistic to identify when the process goes out of control. Whereas the T^2 statistic indicates that a process is out of control, it does not provide information on the specific variable or the set of variables that cause the process to go out of control.

Murphy **[11]** proposed a method to identify the out-of-control variables based on discriminant analysis. He divided the complete set of variables into two subsets and then tried to determine the set that caused the 'out-of-control' signal. Murphy's work was extended by Niaki *et al.* **[13]** and Niaki and Moeinzadeh [**12]. They developed a statistic and an algorithm to estimate population parameters in a cause-selecting problem. Mason *et al.* **[10]** proposed a cause identification procedure by decomposing the T^2 statistic, to assist the user to determine the contribution by each variable based on deviation. The multivariate exponential weighted moving average control charts uses all the observations since the detection of the last special event rather than the only the last observation vector as in Shewhart-type charts. Their advantage is that their average run length is smaller for small shifts in the process mean. Lowry *et al.* [**8,9]** presented a multivariate extension of the exponentially weighted moving average (EWMA) control chart.

In the above-stated research works, the quality characteristics of a product are defined in terms of a numerical value measured in an interval, ratio or absolute scale. In many industrial situations, however, we come across situations where quality has to be defined using linguistic variables using subjective measures like rating on a scale. Bradshaw**[1]** used fuzzy set theory as the basis for interpreting the representation of a graded degree of product conformance with a quality standard. Raz and Wang **[15]** illustrated two approaches for constructing variable control charts based on linguistic data when the product quality is classified as 'perfect','good','poor', etc. Here, representative fuzzy measures are obtained by using any of the four commonly used methods, namely fuzzy mode, the alpha-level fuzzy midrange, the fuzzy median, or the fuzzy average, to construct the control limits. The membership functions defined for the linguistic variables in the above method are chosen arbitrarily and hence decision for process control may change as per the user's choice of values of decision parameters. Probability distributions underlying linguistic data are not considered in this work. Kanagawa *et al.* **[7]** developed control charts for linguistic variables based on probability density functions for the linguistic data in order to control process average and process variability. However, the concept of probability density functions is not appropriate for linguistic data, because such a distribution is often unknown.

Based on fuzzy and probability theory, Taleb and Limam**[16]**, discussed different procedures of constructing control charts for linguistic data, based on fuzzy and probability theory. Taleb and Limam**[17]** developed fuzzy multinomial control chart., where they suggested two approaches for constructing control charts to monitor multivariate attribute processes when data are presented in a linguistic form. They developed two monitoring statistics T_f^2 and W^2 based on fuzzy and probability theories. Cheng **[5]** highlighted a problem associated with the above mentioned approaches. In Taleb and Limam's works **[16, 17]**, the fuzziness assumption is not retained all through the methodological steps.

Cheng **[5]** proposed a novel approach to construct fuzzy control charts by associating fuzzy numbers with experts' scores on quality characteristics. In the current paper, we integrate the concepts forwarded by other researchers and develop a method to construct fuzzy control charts for a product having multiple attribute-type characteristics that are correlated and are measured by experts on a linguistic scale.

3. Methodology

Before formulating the model, we first introduce the symbols used in the model. We develop the model, describing how the subjective scores given by the experts on the values of quality characteristics are converted into fuzzy measures, how they are ranked, and how they are used to detect the out-of-control condition of a process.

3.1 Symbols

We use the following symbols to describe the model:

i	Index for the sample number, $i = 1, 2, \ldots, I$
j	Index for the items in a sample, $j = 1, 2, \ldots, J$
k	Index for the quality characteristic of a product, $k = 1, 2, \ldots, K$
l	Index for the expert, $l = 1, 2, \ldots, L$
q_{ijkl}	Numerical value given by the l^{th} expert for the k^{th} quality characteristic of the j^{th} item in the i^{th} sample.
$\tilde{q}_{ijk}$	Fuzzy value for the k^{th} quality characteristic of the j^{th} item in the i^{th} sample
w_k	Weight assigned to the k^{th} quality characteristic
$\tilde{q}_{ik}$	Average fuzzy value for the k^{th} quality characteristic of the i^{th} sample.
$\tilde{q}_i$	Aggregate fuzzy value of quality of the i^{th} sample.
$\bar{\tilde{q}}$	Fuzzy value for quality to be achieved by process

3.2 Model Description

In this paper we consider a product that has multiple attribute-type characteristics that are correlated and whose values are subjectively estimated by quality experts. Examples of such products are given in Table 1.

Table 1: Products and their Attribute-type characteristics

Product	**Attribute Type**
Tea Leaves	Taste, Liquor and Flavour
Jam/Chocolates	Taste, Texture and Appearance
Cloth	Texture and Type of Stitching
Biscuits	Taste, Appearance and Fragility

We assume that for the purpose of process control, a sample of items is collected from the final products manufactured by a process. A number of quality experts judge the value of each quality characteristic of every item in the sample in an ordinal scale, rating them as good or bad or high or moderate or low, etc.

Consider that the product has K quality characteristics $Q_1, Q_2, \ldots, Q_k$ and that a quality characteristic k is evaluated by the experts on an ordinal scale 1, 2, . . ., G_k. The problem is to judge whether the process is in control on the basis of the experts' subjective estimates of such correlated attribute-type quality characteristics of items in a number of samples. We define q_{ijkl} as the score given by the l^{th} expert on the k^{th} quality characteristic of the j^{th} item belonging to the i^{th} sample ($i = 1, 2, \ldots, I$; $j = 1, 2, \ldots, J$; $k = 1, 2, \ldots, K$; and $l = 1, 2, \ldots, L$).

The scores on a particular quality characteristic, given subjectively by the experts, represent a sample drawn from the possibility distribution of the rating process. We have used the method proposed by Cheng **[5]** to express the aggregate score given by experts on each quality characteristic of an item of a sample as a fuzzy number $\tilde{q}_{ijk}$ and determine its membership function. The details of the method to find $\tilde{q}_{ijk}$ and its membership function are given in Appendix A.

We represent $\tilde{q}_{ijk}$ as a triangular fuzzy number $(\tilde{m}_{ijk}, \tilde{\alpha}_{ijk}, \tilde{\beta}_{ijk})$, where $\tilde{m}_{ijk}$ is the value of $\tilde{q}_{ijk}$ with its membership function taking a value 1, and $\tilde{\alpha}_{ijk}$ and $\tilde{\beta}_{ijk}$ denote the distances of $\tilde{q}_{ijk}$ from $\tilde{m}_{ijk}$ to its left and right extremes respectively with the corresponding membership values as 0.

The average of the k^{th} quality characteristic for all items in a sample i, $\tilde{q}_{ik}$, is another fuzzy number $(\tilde{m}_{ik}, \tilde{\alpha}_{ik}, \tilde{\beta}_{ik})$ which is obtained as follows:

$$\tilde{m}_{ik} = \frac{\sum_{j=1}^{J} \tilde{m}_{ijk}}{J}, \qquad \tilde{\alpha}_{ik} = \frac{\sum_{j=1}^{J} \tilde{\alpha}_{ijk}}{J}, \qquad \text{and} \qquad \tilde{\beta}_{ik} = \frac{\sum_{j=1}^{J} \tilde{\beta}_{ijk}}{J}$$

The final aggregate fuzzy measure $\tilde{q}_i$ of the quality of sample of items aggregated over the K quality characteristics is given as

$$\tilde{q}_i = \sum_{k=1}^{K} w_k . \tilde{q}_{ik} \qquad \text{. . . (1)}$$

where w_k is the weight assigned by the process managers (or experts) to the fuzzy measure of the quality characteristic $\tilde{q}_{ik}$ and is constrained by

$$\sum_{k=1}^{K} w_k = 1 \qquad \text{. . . (2)}$$

As noted earlier, the K quality characteristics of the product may be correlated. To take account of such correlation, the method proposed by Carlsson *et al.* **[4]** is followed here to calculate the weighted sum of the fuzzy measure of the correlated quality characteristic. Appendix B illustrates the use of the method.

3.3. Detection of Process Shift and Variability

3.3.1. Estimation of process mean to be achieved by the process ($\overline{\tilde{q}}$)

To start with, we assume that the process is in statistically stable state. A number of samples of 20 items each are collected and the expert's scores are taken for each quality characteristic in a sample for every item. The fuzzy aggregate measure for the quality of sample, $\overline{\tilde{q}}$, is calculated by the method mentioned in the previous section. This measure gives the fuzzy value that needs to be achieved by items manufactured from the process.

3.3.2. Use of the estimated process mean to construct control chart

Once the process mean is fixed, random samples are taken at regular intervals to detect whether the process is in control or not. For a sample *i*, $\tilde{q}_i$, the fuzzy aggregate measure for the quality of sample obtained from eq. (1) using interactive sum of fuzzy numbers, is compared with $\overline{\tilde{q}}$, the quality that has to be achieved by the process. If $\tilde{q}_i$ is greater than $\overline{\tilde{q}}$, then manufacturing process is considered as in control. If $\tilde{q}_i$ is not greater than $\overline{\tilde{q}}$, then following Prade **[14]**, we test for the proximity of $\tilde{q}_i$ to $\overline{\tilde{q}}$, using the concepts related to possibility theory. The process of ordering of fuzzy numbers and possibility theory is discussed in the next section.

3.4. Ordering of Fuzzy Numbers

The ordering of fuzzy numbers as proposed by Buckley **[2,3]** is discussed below:

Consider two fuzzy numbers $\widetilde{M}$ and $\widetilde{N}$. We write $\widetilde{N} < \widetilde{M}$ if the following holds:

$$\upsilon(\widetilde{N} \leq \widetilde{M}) = 1 \text{ and } \upsilon(\widetilde{M} \leq \widetilde{N}) < \eta \quad \text{. . . (3)}$$

where, for all values of *x* and *y,*

$$\upsilon(\widetilde{M} \leq \widetilde{N}) = \max\{\min(\widetilde{\mathrm{M}}(\mathrm{x}), \widetilde{\mathrm{N}}(\mathrm{y})) \mid \mathrm{x} < \mathrm{y}\}$$

and η is generally taken as 0.8 (some fixed fraction in (0, 1]).

The method of ordering of fuzzy numbers discussed above, is shown in Fig. 1. Fig. 1 shows the $\upsilon(\widetilde{M} \leq \widetilde{N})$ given in eq. (3) to be 0.62 which is less than η (=0.8). Hence, we can say $\widetilde{N} < \widetilde{M}$ since $\upsilon(\widetilde{N} \leq \widetilde{M}) = 1$.

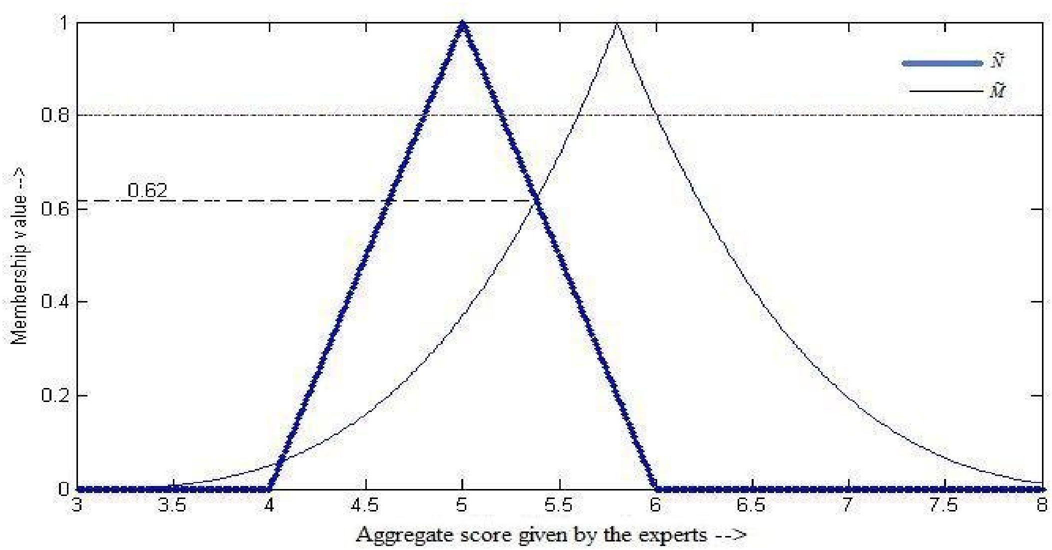


Fig. 1. Plot showing ordering of fuzzy numbers ($\widetilde{\mathrm{N}} < \widetilde{\mathrm{M}}$)

3.5. Possibility Measure and Necessity Measure

The purpose of finding the possibility and necessity measures is to determine how close the sample aggregate value is the process mean to be achieved. If they are very close, then the process is considered to be in control.

Following Prade **[14]**, the possibility measure of the variable $\tilde{q}_i$ satisfying the condition "$\tilde{q}_i$ is $\bar{\tilde{q}}$ "is defined as

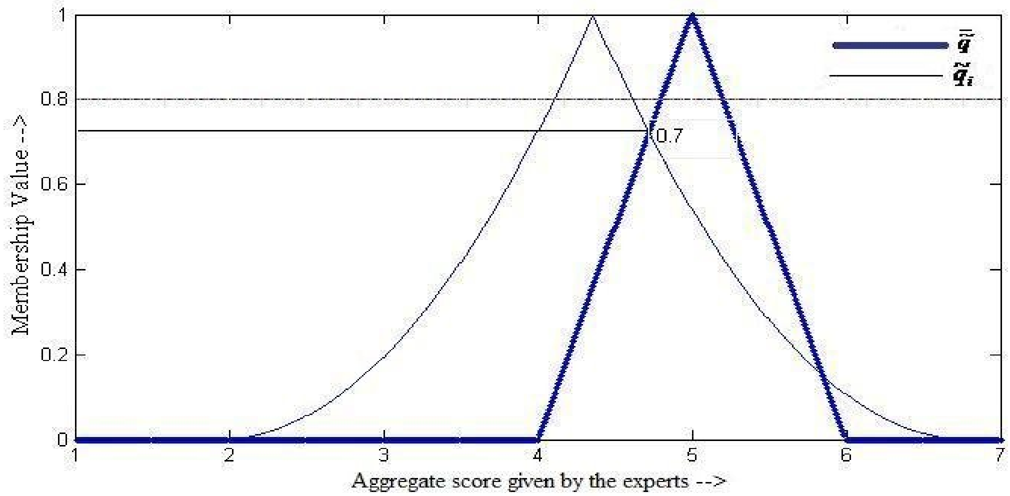
$$Pos(\bar{\tilde{q}} \mid \tilde{q}_i) = \sup_{z \in U} [\min\{\mu_{\bar{\tilde{q}}}(z), \Pi_{\tilde{q}_i}(z)\}] \quad \text{. . . (4)}$$

where $\Pi_{\tilde{q}_i}(z)$ is the possibility distribution of $\tilde{q}_i$, U is the universal set, and z is an element of U.

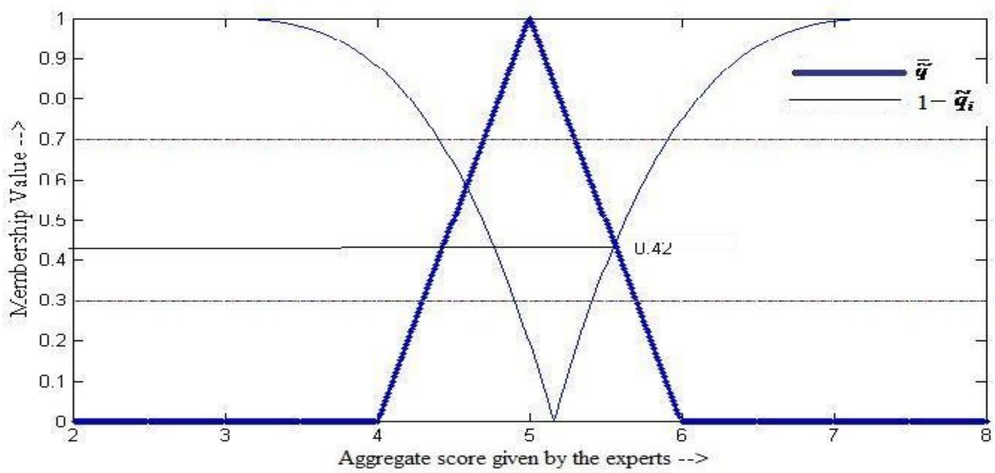
The necessity measure of the variable $\tilde{q}_i$ satisfying the condition "$\tilde{q}_i$ is $\bar{\tilde{q}}$ "is defined as

$$Nec(\bar{\tilde{q}} \mid \tilde{q}_i) = \inf_{z \in U} [\max\{\mu_{\bar{\tilde{q}}}(z), 1 - \Pi_{\tilde{q}_i}(z)\}] \quad \text{. . . (5)}$$

The two conditions given in eq. (4) and eq. (5) are depicted in Fig. 2a and Fig. 2b respectively. Fig. 2a shows the possibility measure between $\tilde{q}_i$ and $\bar{\tilde{q}}$ given in eq. (4) to be 0.7. Fig. 2b represents necessity measure between $\tilde{q}_i$ and $\bar{\tilde{q}}$ given in eq. (5) to be 0.42.



(a) Figure showing possibility measure



(b) Figure showing necessity measure

Fig. 2: plots showing possibility and necessity measure for two fuzzy numbers $\tilde{q}_i$ and $\bar{\tilde{q}}$

3.6. Verification of Sample with In-Control Process Mean

For a process to be considered in control, either of the following two conditions are to be satisfied: (1) $\tilde{q}_i$ should be more than $\bar{\tilde{q}}$ for which the criteria for ordering of fuzzy numbers given in eq. (3) should be satisfied and (2) the possibility measure of fuzzy sample mean, $\tilde{q}_i$

must be greater than or equal to a threshold parameter α, $0 \le \alpha \le 1$, whose value is to be specified by experts. The parameter α reflects the possibility of rejecting a true hypotheses that the process mean is in control. The second condition is equivalent to the following property (Prade **[14]**):

$$\text{If } \operatorname{Pos}(\tilde{\bar{q}} \mid \tilde{q}_i) \ge \alpha, \text{ then } [\tilde{\bar{q}}]_\alpha \cap [\tilde{q}_i]_\alpha \ne \phi \qquad \ldots (6)$$

where $[.]_\alpha$ denotes the α-level set.

But the above measure (eq. (6)) can just indicate the proximity of the mode of the sample mean with that of the in-control process mean. A case may exist when the degree of fuzziness of such a mean can still be far greater than desirable value. So, the necessity measure is used to satisfy the fuzziness aspect. The necessity measure of the fuzzy sample mean $\tilde{q}_i$ must be greater than or equal to threshold value β, $0 \le \beta \le 1$, whose value is to be specified by experts. This condition is equivalent to the following property:

$$\text{If } \operatorname{Nec}(\tilde{\bar{q}} \mid \tilde{q}_i) \ge \beta \text{ , then } [\tilde{\bar{q}}]_\beta \supset [\tilde{q}_i]_{1-\beta} \qquad \ldots (7)$$

The parameter β reflects the possibility of rejecting true hypotheses that the process variability is tolerably small. Usually, α and β values are taken as 0.8 and 0.2 respectively.

3.7. Method for Detecting Quality Characteristic(s) Causing Out-of-Control Point Situation

In this paper, we provide an approach for detecting quality characteristic(s) responsible for an out-of-control situation (if any). If a sample shows out of control condition by the measures given in Section 3.6, i.e., $\tilde{q}_i$ is not close to $\tilde{\bar{q}}$, then if r out of K characteristics are actually out -of-control, then $\tilde{q}_i^{'} = \sum \mathrm{w}_{\mathrm{j}}^{'}.\tilde{q}_{ij}$, summed over the remaining $K - r$ characteristics, should be either greater or close to $\tilde{\bar{q}}$. Here, the weights are increased in such a way that they have their relative importance $w_i^{'}$ same as before and the condition such that $\sum w_i^{'} = 1$. For example, consider three characteristics, Q_1, Q_2 and Q_3. If neither $w_1.\tilde{q}_{i1} + w_2.\tilde{q}_{i2} + w_3.\tilde{q}_{i3}$ is found to be greater than $\tilde{\bar{q}}$ nor their proximity measure is less than the required value, then one suspects the process to be out of control. If the reason for this out-of-control condition is only due to Q_3, then either $w_1^{'}.\tilde{q}_{i1} + w_2^{'}.\tilde{q}_{i2}$ has to be greater than $\tilde{\bar{q}}$ or the proximity between $w_1^{'}.\tilde{q}_{i1} + w_2^{'}.\tilde{q}_{i2}$ and $\tilde{\bar{q}}$ is high.

4. Application of the Method

4.1. A Hypothetical Example

We assume a hypothetical case where the product has three quality characteristics. Every sample drawn has three items, and five experts rate the quality characteristics in a scale of 1 to 10, 1 indicating the lowest value and 10 the highest value of a quality characteristic. Thus, $J = 3$, $K = 3$, and $L=5$. We assume $\tilde{\bar{q}}$, the quality that has to be achieved using the estimation process mentioned in Section 3.3.1, to be given by $\tilde{\bar{q}}$=(5, 1, 1).

On the basis of a randomly drawn sample *i*, we propose a nine-step procedure for judging whether a process is in control or out of control.

Step-1: Identify the three quality characteristics, Q_1, Q_2 and Q_3.

Step-2: Obtain the weight for each quality characteristic from the experts. In this example, we assume the weights for Q_1, Q_2 and Q_3 as $w_1 = 0.3$, $w_2 = 0.4$ and $w_3 = 0.3$ respectively.

Step-3: Obtain score q_{ijkl} for the three quality characteristics from the three experts for all three items in the sample. The assumed ratings given by the experts to the three quality characteristics of each item in the randomly drawn sample are given in the column 3 through column 7 of Table 2.

Step-4: For the j^{th} item in the i^{th} sample, calculate fuzzy number $\widetilde{q}_{ijk}$ for each quality characteristic using the method given in Appendix A. For the example considered, the values are shown in column 8 of Table 2.

Step-5: Calculate the average value of the quality characteristic taken on all items ($\widetilde{q}_{ik}$) using eq. (1). The calculated values for the example are given below:

$\widetilde{q}_{11}$= (5.13, 1.94, 2.99) $\widetilde{q}_{12}$= (5.64, 2.72, 1.94) $\widetilde{q}_{13}$= (5.56, 2.41, 1.95)

Step-6: Obtain the membership function for the aggregate fuzzy number ($\widetilde{q}_i$) for the sample (considered as the first sample, i=1) by using eq. (B.1) and eq. (1). The membership function for $\widetilde{q}_i$ is given as $[\widetilde{q}_1]_\gamma = [5.49 - 2.34 * (1 - \sqrt[3]{\gamma}), 5.49 - 2.34 * (1 - \sqrt[3]{\gamma})]$, where $0 \le \gamma \le 1$

Table 2: Experts' Scores on the Quality Characteristics and Their Equivalent Fuzzy Numbers

Items in the sample	Quality characteristics	Expert 1	Expert 2	Expert 3	Expert 4	Expert 5	Fuzzy Number for the experts' scores
Item-1	Q_1	4	6	7	5	4	(5.07, 1.86, 3.58)
	Q_2	4	5	5	6	6	(5.25, 1.23, 2.01)
	Q_3	6	6	5	7	5	(5.75, 3.05, 1.89)
Item-2	Q_1	4	5	5	5	6	(5, 2.45, 2.45)
	Q_2	5	6	4	8	6	(5.73, 3.37, 1.94)
	Q_3	4	5	6	7	5	(5.32, 1.53, 2.94)
Item-3	Q_1	5	7	5	4	6	(5.32, 1.53, 2.94)
	Q_2	4	7	5	6	7	(5.93, 3.58, 1.87)
	Q_3	6	6	4	5	6	(5.61, 2.64, 1.02)

Step-7: Plot the aggregate fuzzy number ($\tilde{q}_i$) and quality to be achieved by the process ($\overline{\tilde{q}}$). Fig. 3 is a plot of the membership function for $\tilde{q}_i$ and $\overline{\tilde{q}}$.

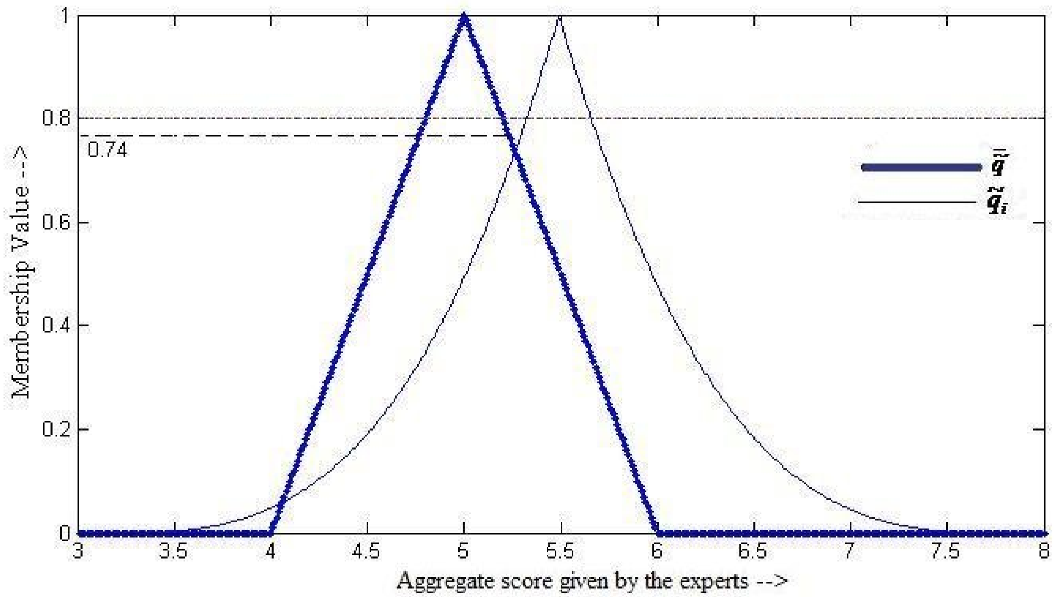


Fig. 3: Plot between sample quality and quality to be achieved

Step-9: Decide whether the process is in control using either ordering of fuzzy numbers or possibility and necessity measures given in Section 3.4 and Section 3.5. In this case, use of eq. (3) leads to comparing the maximum ordinate of point of intersection of two membership functions with the acceptable value of 0.8. In this example, the point of intersection is 0.74, which is less than 0.8, hence the fuzzy number ($\tilde{q}_i$) is considered greater than $\overline{\tilde{q}}$. Hence, there is no need for using the possibility and necessity measures. On the basis of the sample, we cannot consider the process to be out of control.

4.2. An Example of an Out-of-Control Situation

The data for this example are shown in Table 3. Steps 1 through 8 are followed as given in Section 4.1 to obtain the plot (Fig. 4) between the fuzzy number $\tilde{q}_i$ and the quality to be achieved by the process $\overline{\tilde{q}}$. We can clearly see from Fig. 4 that $\overline{\tilde{q}}$ is greater than $\tilde{q}_i$ and hence the sample shows that the process out of control. To find the quality characteristic(s) responsible for the out-of-control situation, the method proposed in Section 3.7 is now followed. We first define the average fuzzy value for the k^{th} quality characteristic of the i^{th} sample, for all values i = 1, 2, and 3:

$\tilde{q}_{11}$ = (3.27,0.73, 3.86) $\tilde{q}_{12}$ = (5.15, 1.75, 3.36) $\tilde{q}_{13}$ = (5.19, 1.44, 2.53)

The membership function for the whole sample is obtained using eq. (1) and eq. (B.1):

$$[\tilde{q}_1]_\gamma = [4.59 - 2.27 * (1 - \sqrt[3]{\gamma}), 4.59 + 2.27 * (1 - \sqrt[3]{\gamma})], \text{ where } 0 \leq \gamma \leq 1$$

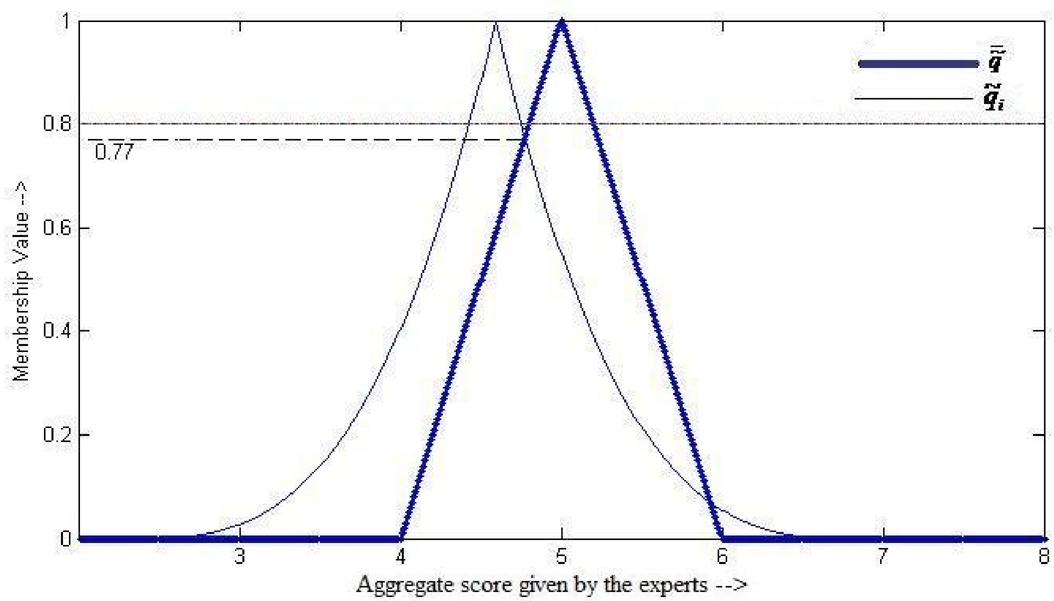


Fig. 4: Membership functions of $\tilde{q}_i$ and $\overline{\tilde{q}}$

Table 3: Experts' Scores on the Quality Characteristics and Their Equivalent Fuzzy Numbers –The Case of Out-of-control Situation

Items in the sample	Quality characteristics	Expert 1	Expert 2	Expert 3	Expert 4	Expert 5	Fuzzy Number as per scores given by experts
Item-1	Q_1	3	3	5	3	5	(3.11, 0.27, 4.6)
	Q_2	4	6	7	5	4	(5.07, 1.86, 3.58)
	Q_3	4	5	5	6	6	(5.25, 1.23, 2.01)
Item-2	Q_1	4	3	3	3	5	(3.61, 1.65, 2.39)
	Q_2	4	5	6	7	5	(5.32, 1.53, 2.94)
	Q_3	4	5	5	6	6	(5.25, 1.23, 2.01)
Item-3	Q_1	3	3	5	3	5	(3.11, 0.27, 4.6)
	Q_2	4	6	7	5	4	(5.07, 1.86, 3.58)
	Q_3	4	6	7	5	4	(5.07, 1.86, 3.58)

Since there are three quality characteristics, we have to test for all possible combinations of two quality characteristics, taken two at a time. In this case, we have to test for the in-control situations for the combinations Q_1 and Q_2, Q_1 and Q_3, and Q_2 and Q_3. Weights have to be changed, retaining their relative importance originally set, so that they also sum up to 1.

4.2.1. Check for out-of-control characteristics

we have

$$\tilde{q}_{11} = (3.27, 0.73, 3.86) \quad \text{and} \quad \tilde{q}_{12} = (5.15, 1.75, 3.36)$$

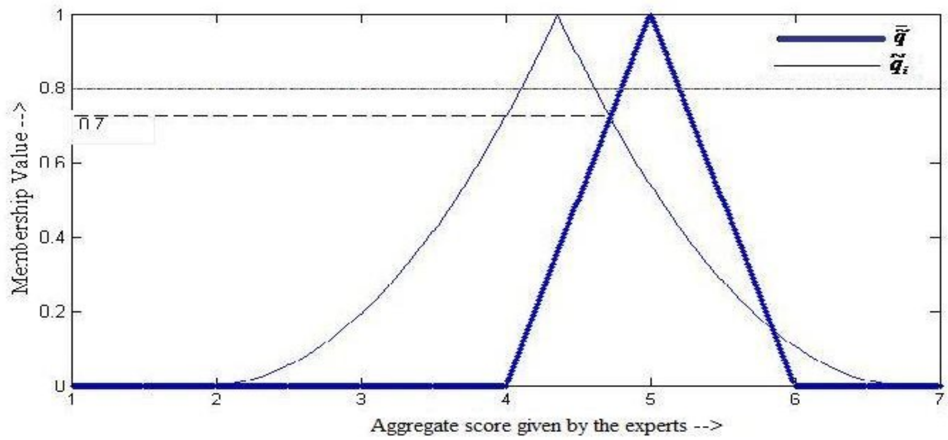
The new weights for Q_1 and Q_2 are 0.42 (=0.3/ (0.3+0.4)) and 0.58 (=0.4/ (0.3+0.4)) respectively. The final aggregate membership function is obtained as

$$[\tilde{q}_1]_\gamma = [4.36 - 2.43 * (1 - \sqrt[3]{\gamma}), 4.36 + 2.43 * (1 - \sqrt[3]{\gamma})], \text{ where } 0 \le \gamma \le 1$$

We can see in Fig. 5a that this measure is less than the required quality. It represents the aggregate fuzzy value for the sample quality considering only the quality characteristics Q_1 and Q_2. Fig. 5a shows that this aggregate fuzzy sample value is less than the desirable fuzzy value.

Similarly, Fig. 5b represents the aggregate fuzzy value for the sample quality considering only the quality characteristics Q_1 and Q_3, the fuzzy sample quality coming out to be less than the desirable fuzzy value.

(a) **Q_1 and Q_2**



(b) **Q_1 and Q_3**

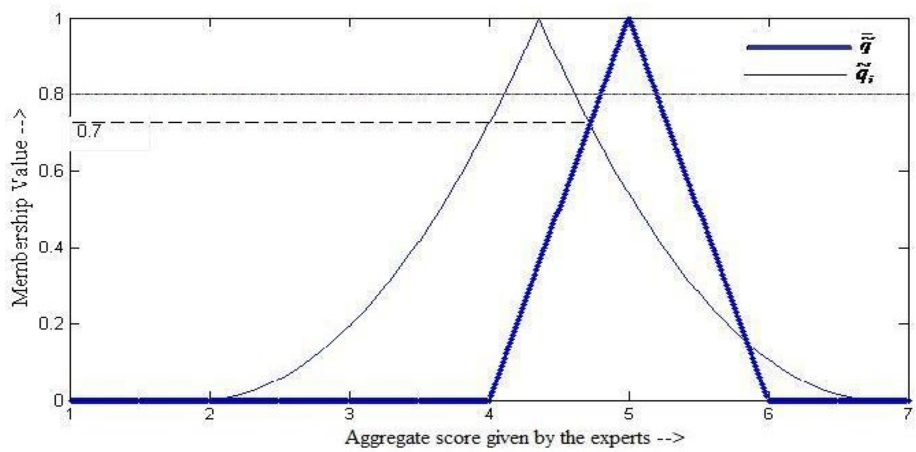
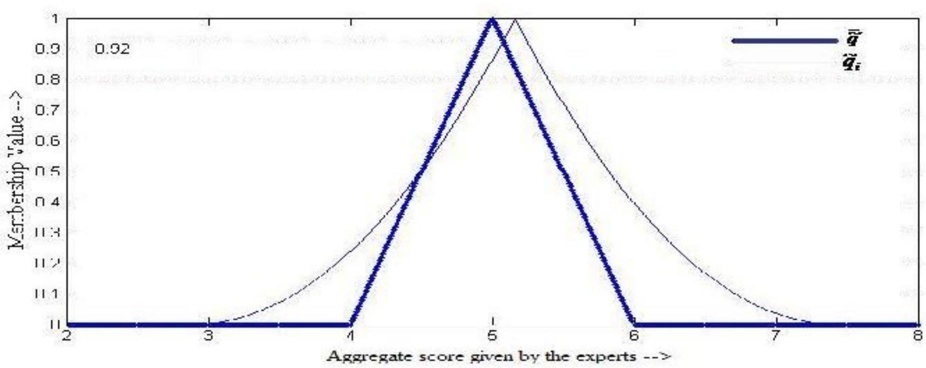


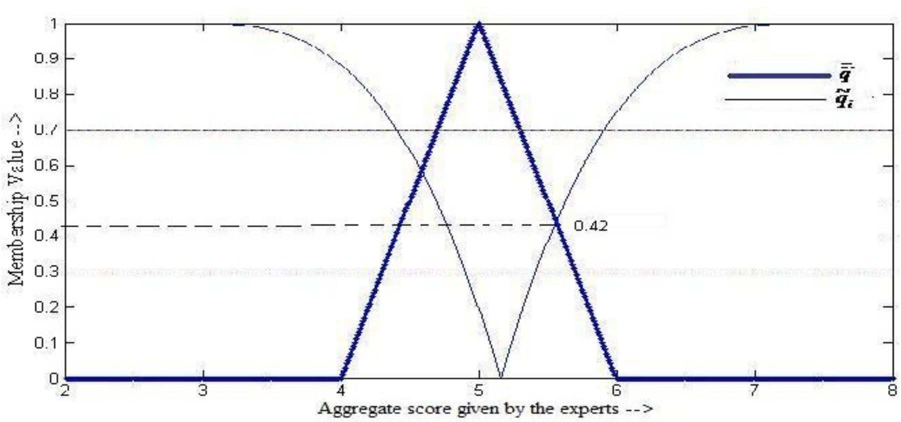
Fig. 5: Plots for showing out-of-control situation considering only two characteristics.

We conclude that when judged on the basis of sample values of quality characteristics Q_1 and Q_2, and Q_1 and Q_3 respectively, the process is considered to be out of control, indicating that either all the characteristics Q_1, Q_2, and Q_3 or only Q_1 is responsible to out-of-control condition of the process. To come to a firm decision, we have to make further investigation considering only the quality characteristics Q_2 and Q_3.

For the quality characteristics Q_2 and Q_3, we have plots as shown in Fig. 6a and Fig. 6b. Fig. 6a represents the possibility measure between the sample value considering only Q_2 and Q_3. The possibility measure comes out to be 0.92 greater than α (= 0.8). Fig. 6b represents the necessity measure between the sample value considering only Q_2 and Q_3. Here the necessity measure comes out to be 0.42 greater than β (=0.3). From these two plots, we conclude that when judged from the observation on the quality characteristics Q_2 and Q_3, the process is in-control.



(a) Possibility measure



(b) Necessity Measure

Fig. 6: Plots for showing in-control situation considering only Q_2 and Q_3 characteristics.

The only combination showing an in-control is Q_2 and $Q_{3,}$ and hence, we can conclude that Q_1 is the only quality characteristic that contributes to the process going out of control.

In the Sections 4.3.1 and 4.3.2, we illustrated how to find out-of-control situations of the process for a given sample. In the next section, we show how to construct fuzzy control charts for a series of samples taken at different time points. Two charts can be drawn by using: (1) the possibility measure using eq. (6) and (2) the necessity measure using eq. (7).

4.3. Fuzzy Control Charts Using Possibility and Necessity Measures

One requires to construct two fuzzy control charts, one using the possibility measure and the other using the necessity measure. Each of the two fuzzy control charts has the sample numbers in its abscissa. The fuzzy control chart using the possibility measure has α-level set of the process mean with the lower and the upper values of the set indicated by two horizontal lines. The α-level sets of the aggregate fuzzy sample values are plotted on this chart in the form of vertical lines. A fuzzy control chart using the necessity measure has β-level set of the process mean with the lower and the upper values of the set indicated by two horizontal lines. The $(1 - \beta)$-level sets of the aggregate fuzzy sample values are plotted on this chart in the form of vertical lines.

To illustrate the construction of a control chart, we consider five samples. Table 4 gives the average fuzzy value for the three quality characteristics of each sample.

Table 4: Average Fuzzy Number for the Experts' Scores for Three Quality Characteristics

	$\widetilde{q}_{11}$	$\widetilde{q}_{12}$	$\widetilde{q}_{13}$
Sample-1	(5.73,2.28,3.16)	(5.07,1.86,4.29)	(5.73,3.38,1.92)
Sample-2	(5.73,2.28,3.16)	(5.73,2.28,3.16)	(5.73,2.28,3.16)
Sample-3	(5.25,1.23,2.01)	(5.25,1.23,2.01)	(5.25,1.23,2.01)
Sample-4	(5.25,1.2,2.31)	(5,2.45, 2.45)	(5,2.45, 2.45)
Sample-5	(3.75,2.02,1.21)	(5,0.14,2.21)	(5,0.14,2.21)

To construct the control chart using the possibility measure, we use the values of $\alpha = 0.8$, as mentioned in Section 3.6. For this value α, the corresponding α-level set values of desired quality of the process mean is {4.8, 5.2}. Similarly, we take $\beta = 0.3$ for the control chart using the necessity measure. And the β-level set values for desired quality of the process mean is {4.3, 5.7}. Thus

$$\left[\overline{\widetilde{q}}\right]_{\alpha} = \{4.8,\ 5.2\}\ for\ \alpha = 0.8 \text{ and } \left[\overline{\widetilde{q}}\right]_{\beta} = \{4.3,\ 5.7\}\ for\ \beta = 0.3 \qquad \ldots (7)$$

For plotting the sample values, we first obtain the α-level set of $\widetilde{q}_i$ for the *i*th sample and plot it as a vertical in the chart for possibility measure. Similarly, we obtain the (1-β)-level set values of $\widetilde{q}_i$ for the *i*th sample and plot it as a verticle line in the chart for necessity measure. These α and (1-β)-level set values can be read off from figures depicting possibility and necessity measures for each sample. Fig. 7a and Fig. 7b depict the possibility, necessity measures, and **α** and (1-β)-level set values for sample 1. The α –level set for this sample is {5.4, 5.6} and (1-β)-level set is {5.4, 5.6}.

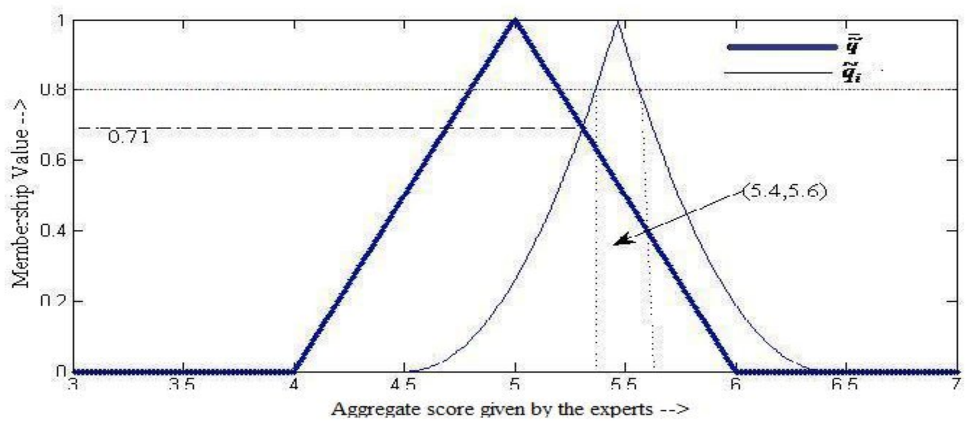
The α and (1-β)-level set values for other sample are read off similarly from Fig. 8a and Fig. 8b, Fig. 9a and Fig. 9b, Fig. 10a and Fig. 10, and Fig. 11a and Fig. 11b that respectively depict the possibility and necessity measures for the remaining four samples.

Sample 2: {5.65, 5.9} and {5.6, 5.9}

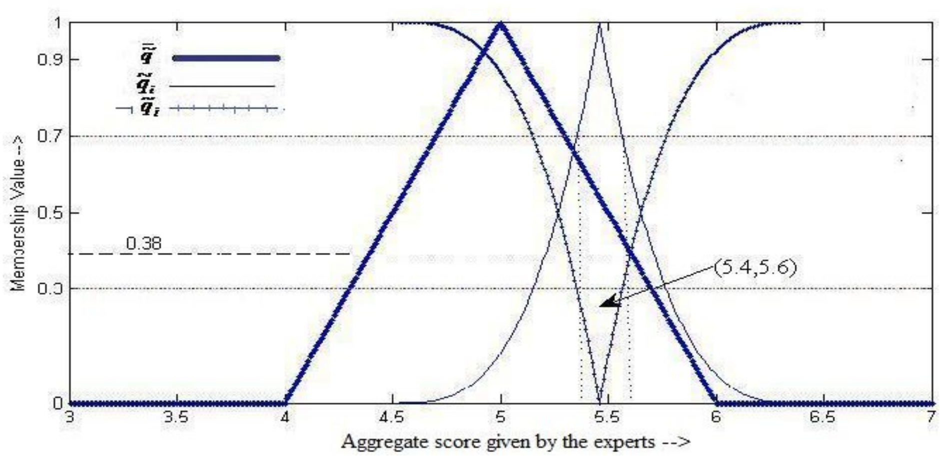
Sample 3: {5.2, 5.3} and {5.2, 5.4}

Sample 4: {5, 5.2} and {5, 5.25}

Sample 5: {4.6, 4.7} and {4.55, 4.7}

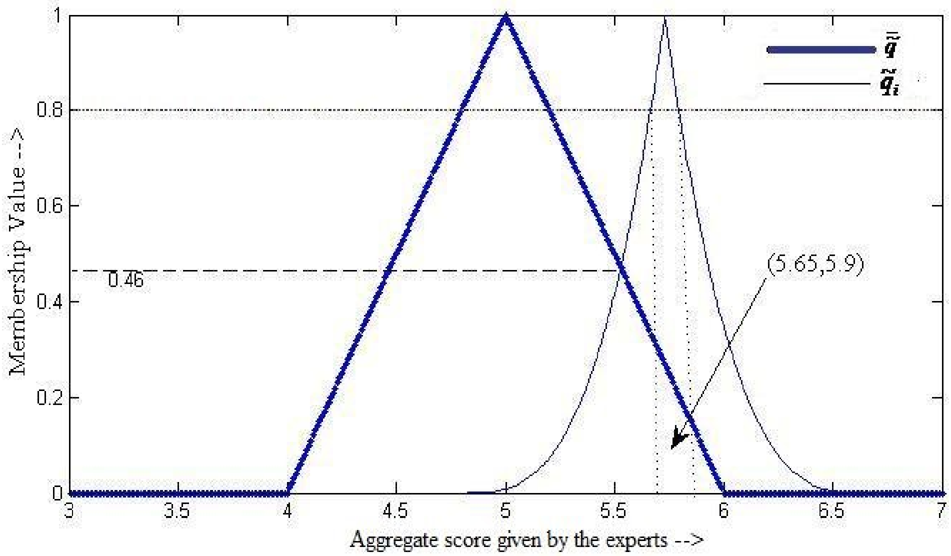


(a) Possibility measure for Sample 1

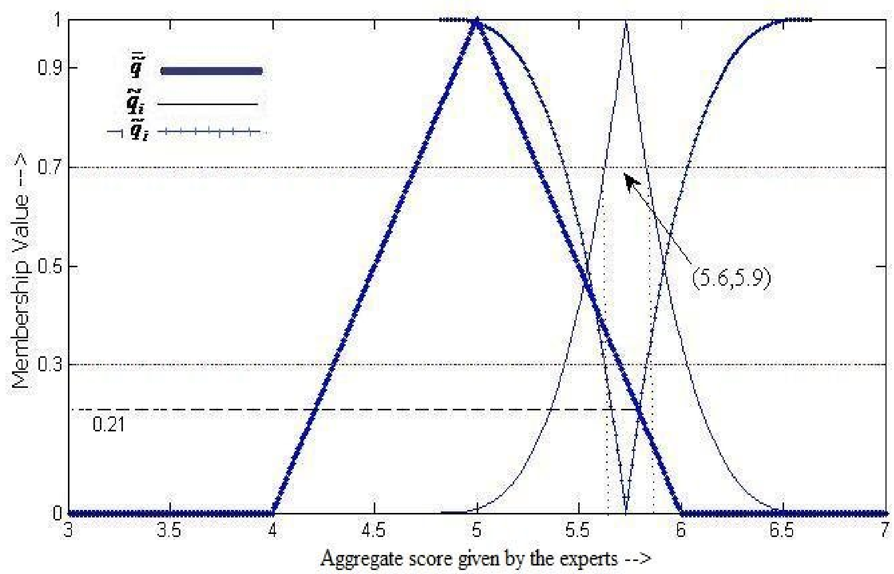


(b) Necessity measure for sample 1

Fig. 7.Possibility measure and Necessity measure of sample 1

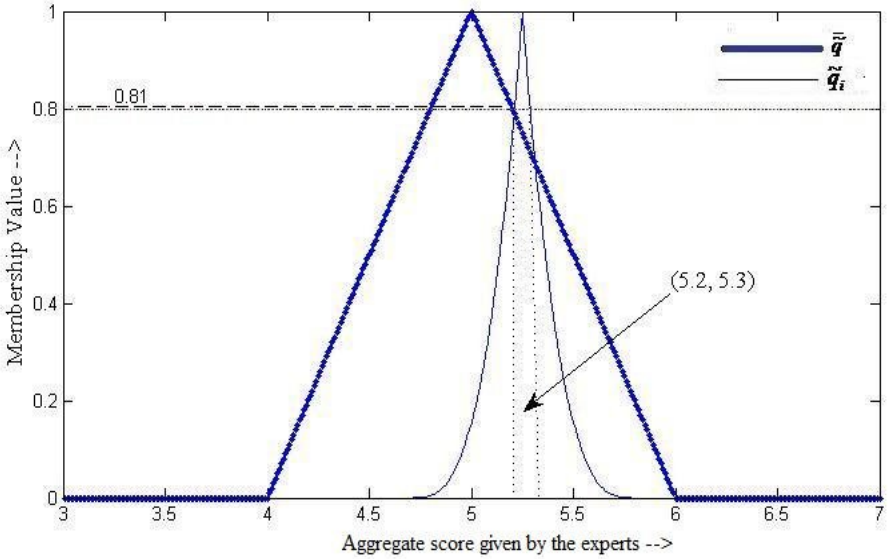


(a) Possibility measure for sample 2

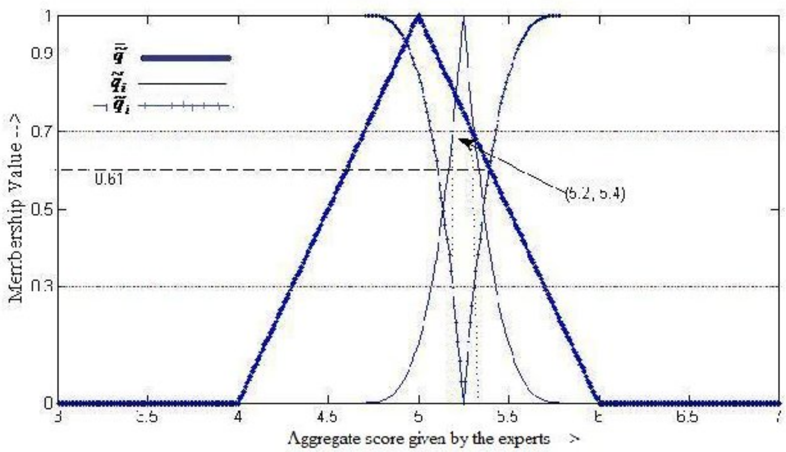


(b) Necessity measure for sample 2

Fig. 8. Possibility measure and Necessity measure of sample 2

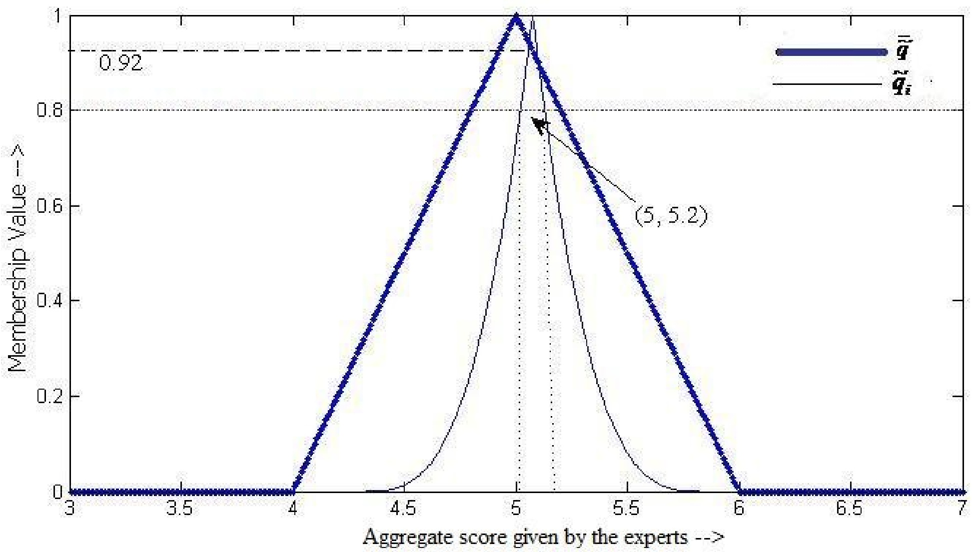


(a) Possibility measure for sample 3

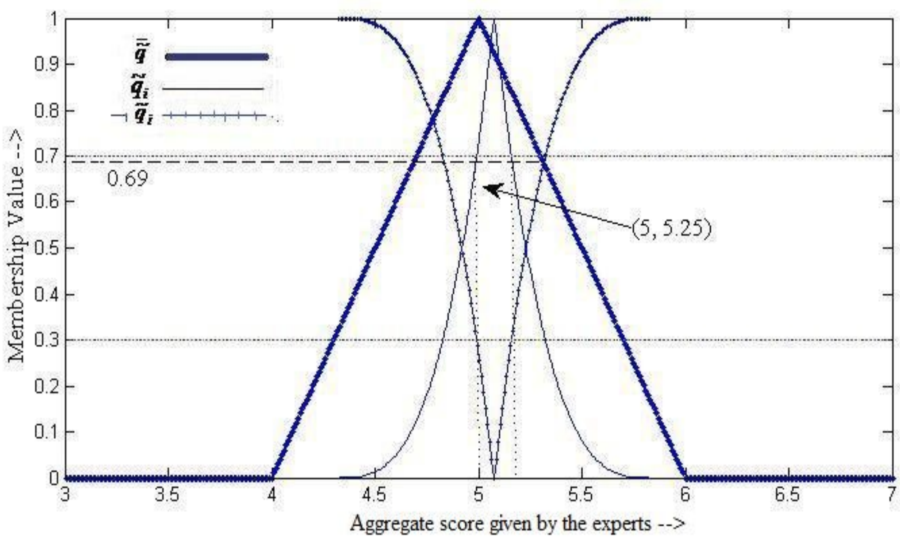


(b) Necessity measure for sample 3

Fig. 9. Possibility measure and Necessity measure of sample 3

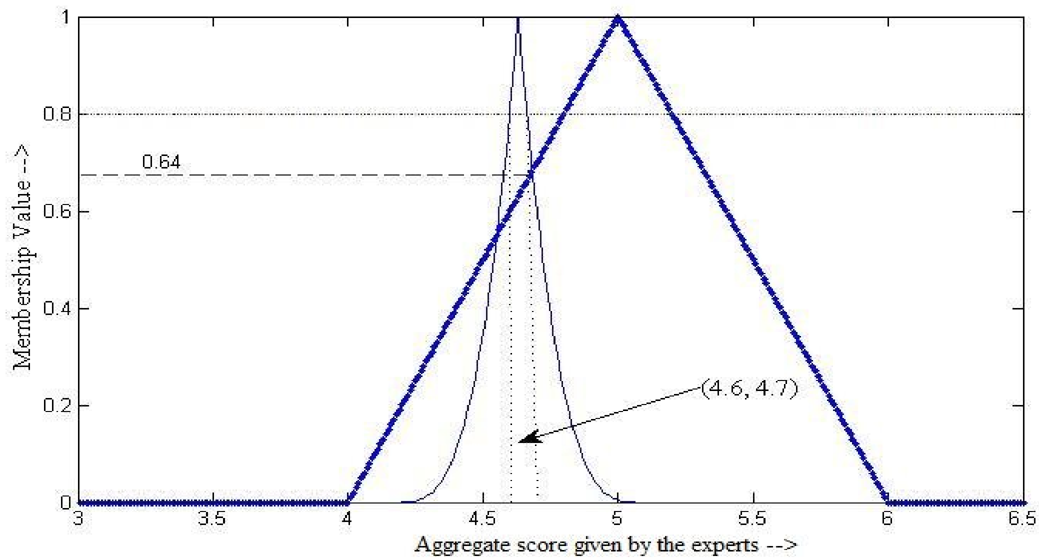


(b) Possibility measure for sample 4

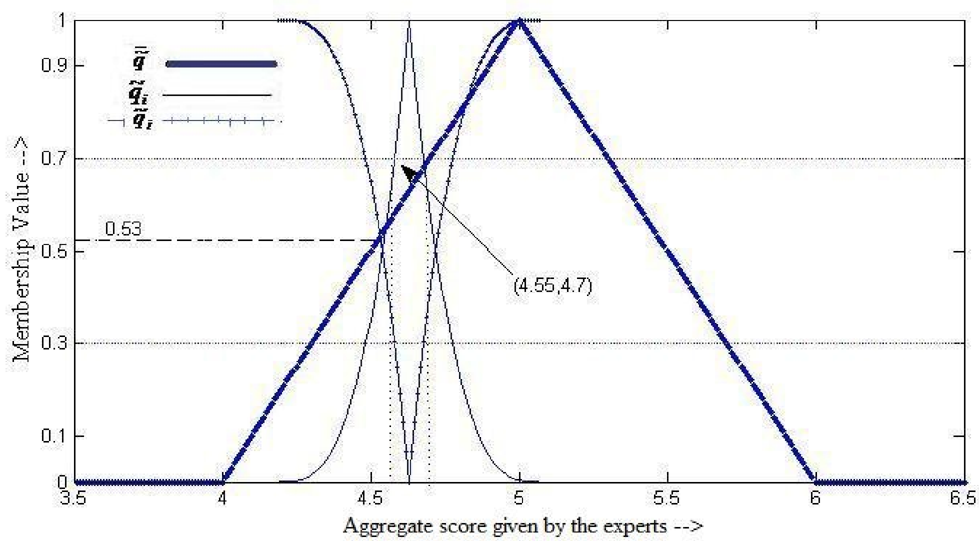


(b) Necessity measure for sample 4

Fig.10. Possibility measure and Necessity measure of sample 4



(a) Possibility measure for sample 5



(b) Necessity measure of sample 5

Fig. 11. Possibility measure and Necessity measure of sample 5

To draw the fuzzy control charts, as mentioned earlier in this section, we need to use properties defined in eq. (6) and eq. (7). Fig. 12 represents the fuzzy control chart using possibility measure. The two horizontal lines in figure show the lower and upper bounds of the α– level set of $\overline{\tilde{q}}$ (here $\alpha = 0.8$) .Vertical line at every sample number point denotes the α- level set of $\tilde{q}_i$. From samples 1 and 2, we consider process to be in-control, as they were decided based on ordering of fuzzy numbers rather than possibility and necessity measures.

In control chart using possibility measure, Out-of-control situation occurs when intersection between α-level set of $\overline{\tilde{q}}$ and α-level set of $\tilde{q}_i$ is a null-set. In Fig. 12, we can clearly see that (leaving samples 1 and 2), only sample 5 gives an out-of-control signal and hence, we conclude that process may go out-of-control.

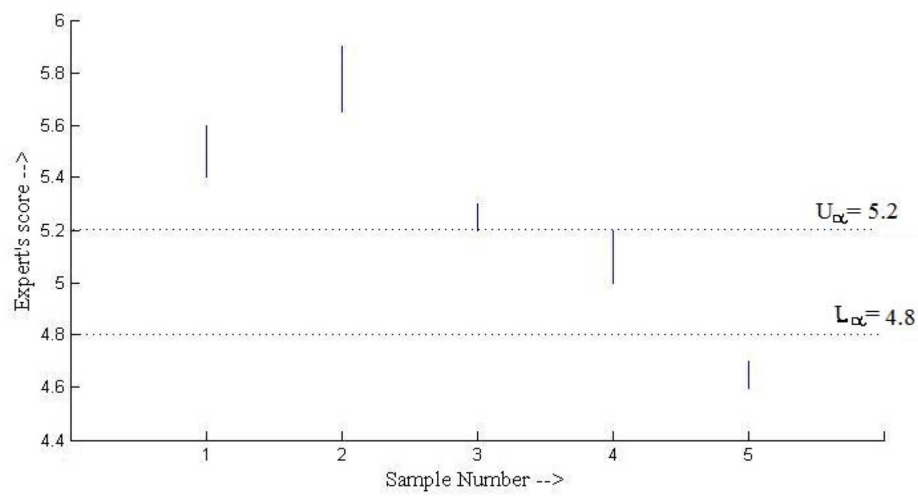


Fig. 12: Fuzzy control chart using possibility measure

Fig. 13 represents the fuzzy control chart using the necessity measure. The two horizontal lines in Fig. 13 show the lower and upper bounds of the β-level set of $\overline{\tilde{q}}$ (here β = 0.3). The vertical line at every sample point denotes the (1-β)-level set of $\tilde{q}_i$. An out-of-control situation occurs when (1-β)-level set of $\tilde{q}_i$ is not a proper subset of β-level set of $\overline{\tilde{q}}$. In this case (leaving samples 1 and 2), we have no sample of this type. Hence, we can conclude all samples show the process to be in stable state with necessity measure.

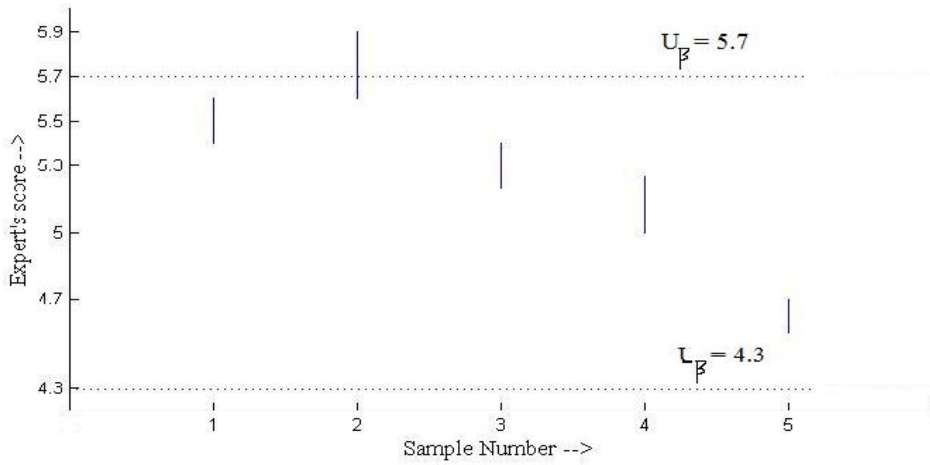


Fig. 13: Fuzzy control chart using necessity measure

Because the two control charts lead to two different conclusions, we need to check both possibility and necessity measures in deciding state of the process, instead of taking decision on the basis of only chart. If both indicate an in-control condition, then the process can be considered to be in control. If, on the other hand, one or both of the charts show an out-of-control signal, then we conclude that process is out-of-control. In the above example, we can conclude process may go out-of-control because possibility charts gives an out-of-control signal.

5. Summary, Salient Features, and Limitations

This paper proposes a method for designing control charts for multi attribute quality characteristics that are correlated and are judged subjectively by quality experts. The method is based upon fuzzy set theory and builds upon the earlier works of Cheng **[5]** and Prade **[14]**. The method makes use of addition of interactive fuzzy numbers proposed by Carlson **[4]** to take care of correlation between individual quality characteristics. The salient features of the work presented are the following:

i) The paper combines the contributions of various past researchers and presents a comprehensively derived model for design of control charts for multi-attribute quality characteristics.

ii) Ordering of fuzzy numbers and the concepts of possibility and necessity measures are used to identify the quality characteristics that cause a process to go out of control.

The limitations of the work are given below:

i) If the number of quality characteristics that cause the process to go out of control is high (3 or more), then it is better to make computer-based calculation to save time and effort.

ii) The choice of weights by the experts plays an important role in the evaluation of the aggregate fuzzy value of the sample quality. This choice has to be decided carefully as it may change the inference drawn about the control condition of the process.

iii) In our paper, we assumed all the quality characteristics to be correlated. Our method is efficient if this assumption holds. If all quality characteristics are not correlated, then the method can still be used. To handle such a case, one needs to separate out those characteristics that are independent. For these characteristics, one needs to draw separate control charts. For the remaining correlated quality characteristics, one can use the method proposed in this paper.

iv) The sensitivity of the fuzzy control charts depends on the chosen values of α, β, and the weights for the quality characteristics.

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Appendix

A. Fuzzy Number Construction

A method for constructing fuzzy number from a series of observations is given in the following steps:

Step-1: Let the observations for a quality characteristic given by n-experts be $g_1, g_2, \ldots g_n$ given on a scale of 0 to G. A relative distance matrix $D=[d_{ij}]_{nxn}$ where $d_{ij}=|g_i-g_j|$ is evaluated.

Step-2: The average of relative distance for each g_i's is calculated by $\bar{d}_i = \sum_{j=1}^{n} d_{ij}/(n-1)$ .This average distance is used to measure the proximity of g_i to center of all the ratings.

Step-3: A pair-wise comparison matrix $P==[p_{ij}]_{(nxn)}$, where $p_{ij} = \frac{\bar{d}_j}{\bar{d}_i}$

Step-4: Evaluate weights by weight determination method of Saaty(1980) as

$$w_j = \frac{1}{\sum_{i=1}^{\backslash n} p_{ij}}, \quad j=1,2,\ldots n. \text{ where } \sum_{i=1}^{n} w_i = 1$$

Step-5: The importance degree w_i represents the weight to be associated with g_i when estimating the mode of the fuzzy number. Mode is given by

$$m = \sum_{i=1}^{n} w_i g_i$$

Step-6: For estimation of spreads, the variance is calculated from the sample of ratings as

$$\widehat{s}^2 = \frac{1}{2}\{\frac{\sum_{i \in \arg(g_i < m)} w_i (m - g_i)^2}{\sum_{i \in \arg(g_i < m)} w_i} + \frac{\sum_{i \in \arg(g_i > m)} w_i (m - g_i)^2}{\sum_{i \in \arg(g_i > m)} w_i}\}$$

Step-7: Ratio of left spread to right spread is calculated by

$$\widehat{\rho} = \frac{m - g^l}{g^r - m} \qquad \text{where} \qquad g^l = \frac{\sum_{i \in \arg(g_i < m)} w_i g_i}{\sum_{i \in \arg(g_i < m)} w_i}$$

$$\text{and} \qquad g^r = \frac{\sum_{i \in \arg(g_i < m)} w_i g_i}{\sum_{i \in \arg(g_i < m)} w_i}$$

Step-8: The spreads of the fuzzy number being constructed is given by

$$a = m - \sqrt{\frac{12\rho^2 s^2}{1 + \rho^2}}$$

$$\text{and} \qquad b = m - \sqrt{\frac{12 s^2}{1 + \rho^2}}$$

Hence the triangular fuzzy number denoted by (a,m,b) with parameter m,a,b is given by

$$t(g : a, m, b) = \begin{cases} 1 - \dfrac{m - g}{m - a}, & a \le g \le m, \\ 1 - \dfrac{g - m}{b - m}, & m < g \le b, \\ 0, & \text{elsewhere}, \end{cases}$$

where $0 \le g \le G$.

B. Addition of Interactive Fuzzy Numbers

The quality characteristics considered in the model can have interaction between the values given by the experts. This has been addressed by the theory of interactive addition of fuzzy numbers.

A Fuzzy Set A in R is said to a γ-level set of a fuzzy set A in R^m is defined by $[A]^\gamma=\{x \varepsilon R^m: A(x)\geq\gamma\}$ if $\gamma>0$ and closure of support of A is denoted by $[A]^\gamma=cl\{ x \epsilon R^m: A(x)>\gamma\}$ if $\gamma=0$.

A fuzzy set B in R^m is said to be a joint possibility distribution of fuzzy numbers $A_i \epsilon F$ *(*family of fuzzy numbers), $i=1,2,..m$, if it satisfies the relationship

$$\max_{x_j \in R, j\neq i} B(x_1, x_2, ...x_m) = A_i(x_i), \qquad \forall x_i \in R, i = 1,..m$$

A_i is called the i^{th} marginal possibility distribution of B and projection of B on the i^{th} axis is A_i for i=1,2,..m.

Fuzzy Numbers $A_i \epsilon F$, i=1,2,..m, are said to be non-interactive if their possibility distribution is given by

$$B(x_{1,....}x_m)=\min\{A_1(x_1),....., A_m(x_m)\}, \text{ for all } x_{1,}...x_m \epsilon R,\ i=1,...m.$$

This mean any change in membership function of A does not affect the second marginal possibility distribution and vice versa. A and B are said to be interactive if they cannot take their values independently of each other. A results which can be used for this current scenario is given below:

Result: Let $\{ A_{1,\dots} A_n \} \epsilon\, F$ be a set of Fuzzy numbers. Let C be their joint possibility distribution, and let f: $R^n \rightarrow R$ be a continuous function. Then,

$$[f_C(A_{1,\dots,}A_n)]^{\gamma}=[f(C)]^{\gamma}=f([C]^{\gamma}) \quad \textit{for all } \gamma\epsilon[0,1].$$

Example-1: The above result can be used for case of product having three quality characteristics Q_1, Q_2 and Q_3.

A fuzzy number is denoted by A=(a, $_1$, $_1$) , B=(b, $_2$, $_2$) and C=(x, $_3$, $_3$).

$[A]^{\gamma}=[a-(1-\gamma)\ _1,a+(1-\gamma)\ _1]$ $[B]^{\gamma}=[b-(1-\gamma)\ _2,b+(1-\gamma)\ _2]$ and

$[C]^{\gamma}=[c-(1-\gamma)\alpha_3,c+(1-\gamma)\ _3]$

$$1-\frac{|x-a|}{\alpha_1}>\gamma \qquad 1-\frac{|x-a|}{\beta_1}>\gamma$$
$$1-\frac{|y-b|}{\alpha_2}>\gamma \qquad 1-\frac{|y-b|}{\beta_2}>\gamma$$
$$1-\frac{|z-c|}{\alpha_3}>\gamma \qquad 1-\frac{|z-c|}{\beta_3}>\gamma$$

Let the joint possibility distribution P be defined by the product t-norm,

i.e., P(x,y,z) = A(x).B(y).C(z)

Let $f(x_{1,}x_2)= x+y+z$ be an addition operator on R^3. Then

$$[P]^{\gamma}= cl\{(x,y,z)\epsilon R^3|P(x,y,z)>\gamma\}$$

Then $[A+B+C]^{\gamma}= cl\{(x,y,z)\epsilon R^3|P(x,y,z)>\gamma\}$

Since $a+b+c-|x-a|-|y-b|-|z-c|\leq x+y+z\leq a+b+c+|x-a|+|y-b|+|z-c|$As , Arithmetic Mean ≥Geometric Mean (AM≥GM)

$$\frac{(\alpha_1 - |x-a|) + (\alpha_2 - |y-b|) + (\alpha_3 - |z-c|)}{3}) \geq \sqrt[3]{(\alpha_1 - |x-a|) * (\alpha_2 - |y-b|) * (\alpha_3 - |z-c|)}$$

$$\frac{|x-a| + |y-b| + |z-c|}{3} \geq \{\frac{\alpha_1 + \alpha_2 + \alpha_3}{3} - \sqrt[3]{\alpha_1 \alpha_2 \alpha_3 \gamma}\}$$

taking $\alpha_1 = \alpha_2 = \alpha_3 = avg\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} = \alpha$

$$\Rightarrow |x-a| + |y-b| + |z-c| \geq 3\alpha(1 - \sqrt[3]{\gamma})$$

Hence $[A+B+C]^{\gamma} = [a+b+c - 3\alpha(1-\sqrt[3]{\gamma}), a+b+c + 3\alpha(1-\sqrt[3]{\gamma})]$ . . . (B.1)